PROBABILITY THEORY I - EXERCISE SET 1

Exercise 1. Let F_X be the probability distribution function of a scalar random variable X. Compute the probability distribution functions of the random variables X^3 ,$-X$, X^2.

Exercise 2. Let F_X be the probability distribution function of a scalar random variable X. Show that F_X is continuous at $a \in \mathbb{R}$ if and only if $\mathbb{P}(X = a) = 0$.

Exercise 3. Let $\Omega = \{1, 2, 3, 4\}$ and $\mathcal{F} = 2^{\Omega}$. Construct a subset $\mathcal{A} \subset \mathcal{F}$ and two different probability measures $\mu_{1,2}$ on $\mathcal{F}$ such that

- $\sigma(\mathcal{A}) = \mathcal{F}$;
- $\mu_1(A) = \mu_2(A)$ for every $A \in \mathcal{A}$.

Exercise 4. (Atomic σ-algebras) Let Ω be a set partitioned into a disjoint union of its subsets: $\Omega = \sqcup_{t \in T} \Omega_t$, where T is an arbitrary index set. Prove that

$$\{\cup_{t \in T'} \Omega_t : T' \subset T\}$$

is a σ-algebra.

Exercise 5. Prove that all σ-algebras on a finite or countable set are atomic.

Exercise 6. Prove that there are exactly 203 σ-algebras on the set $\{1, 2, 3, 4, 5, 6\}$.

Exercise 7. Prove that in the Alice-Bob game (see Section 1.1 of the notes), the probability for Alice to win is $\frac{2}{3}$ (Hint: generalize the game to the case when Alice starts with $10 + k$ euros and Bob starts with $5 - k$ euros,$-10 \leq k \leq 5$).

Exercise 8. ("Uniform probability measure on integers" does not exist) Let $\mathcal{A} \subset 2^{\mathbb{N}}$ be the set for all subsets A of $\mathbb{N}$ that have a *density*, that is, such that the limit

$$\lim_{n \to \infty} \frac{|A \cap \{1, \ldots, n\}|}{n}$$

exists. Give an example of $A \subset \mathbb{N}$ such that $A \notin \mathcal{A}$. Show by examples that $A, B \in \mathcal{A}$ does not imply $A \cap B \in \mathcal{A}$.