Probability theory I - exercise set 0.

You need not return the solutions for this set. Most of the facts are assumed to be familiar from the Measure theory course; if they are not, then you should do the exercises.

Exercise 1. Let μ be a measure. Prove *monotonicity* of μ: if $A \subset B$, then $\mu(A) \le \mu(B)$, whenever both sides make sense.

Exercise 2. Let μ be a measure. Prove the *union bound*:

$$\mu(\cup_{i=1}^{\infty} A_i) \le \sum_{i=1}^{\infty} \mu(A_i),$$

whenever the right-hand side makes sense.

Exercise 3. Let $\mathcal{F}$ be a σ-algebra and $\mu : \mathcal{F} \to \mathbb{R}_{\ge 0} \cup \{+\infty\}$ be a finitely additive function, that is, $\mu(\emptyset) = 0$ and $\mu(A \sqcup B) = \mu(A) + \mu(B)$ for all (disjoint) $A, B \in \mathcal{F}$. Prove that the following are equivalent:

(1) μ is a measure;
(2) μ is *lower semicontinuous,* that is, if $A_1 \subset A_2 \subset \dots$ are in $\mathcal{F}$, then $\mu(\cup_{i=1}^{\infty} A_i) = \lim_{i\to\infty} \mu(A_i)$.

Prove that if μ is *finite*, then any of these conditions are equivalent to *upper semi-continuity*: if $A_1 \supset A_2 \supset \dots$ are in $\mathcal{F}$, then $\mu(\cap_{i=1}^{\infty} A_i) = \lim_{i\to\infty} \mu(A_i)$.

Exercise 4. Show by example that an infinite measure need not be upper semi-continuous.

Exercise 5. Let $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$ be measurable spaces. Suppose $f : \Omega_1 \to \Omega_2$ is a map such that $f^{-1}(A) \in \mathcal{F}_1$ for every $A \in \mathcal{A}$, where $\mathcal{A} \subset \mathcal{F}_2$ and $\sigma(\mathcal{A}) = \mathcal{F}_2$. Prove that f is $\mathcal{F}_1$-to-$\mathcal{F}_2$ measurable.

Exercise 6. Prove that any continuous map $f : \Omega_1 \to \Omega_2$ between topological spaces is $\mathcal{B}(\Omega_1)$-to-$\mathcal{B}(\Omega_2)$ measurable.

Exercise 7. Prove that $\mathcal{B}(\mathbb{R}) = \sigma(\{(-\infty; a] : a \in \mathbb{R}\})$. Conclude that a function $f : \Omega \to \mathbb{R}$ is $\mathcal{F}$-to-$\mathcal{B}(\mathbb{R})$ measurable if and only if $f^{-1}((-\infty, a])$ is $\mathcal{F}$-measurable for every $a \in \mathbb{R}$.

Exercise 8. (measures cannot be pulled back by maps) Let Ω' be a set, $(\Omega; \mathcal{F}; \mu)$ a measure space, and let $f : \Omega' \to \Omega$ be a surjective map. Show by examples that $\{A \subset \Omega' : f(A) \in \mathcal{F}'\}$ is not necessarily a σ-algebra, and even when it is, $\mu'(A) := \mu(f(A))$ is not necessarily a measure.