Probability theory I - exercise set VII

Exercise 1. Let X be a $\frac{1}{2}$-Bernoulli random variable ($\mathbb{P}(X=1)=\mathbb{P}(X=-1)=\frac{1}{2}$) and Y be a standard Gaussian independent of X. Prove that XY and Y are uncorrelated standard Gaussians, but that they are not independent.

Exercise 2. Let $\alpha \in (0;2)$, and let $X_1, X_2, \dots$ be i. i. d. random variables with density $f_{\alpha,0}$ (see Exercise 5 below). Denote $Y_n = \max(X_1, \dots, X_n)$. Prove that the sequence $n^{-\frac{1}{\alpha}} Y_n$ converges in distribution, and identify its limit.

Exercise 3. Let $X_1, X_2, \dots$ be centered i. i. d. such that $\mathbb{E}X_1^2 < \infty$. Prove that

$$n^{-\frac{1}{2}} \max(X_1, \dots, X_n) \xrightarrow{\mathcal{P}} 0.$$

Exercise 4. (A CLT for non-identically-distributed random variables) Let $X_1, X_2, \dots$ be indepedent centered random variables and denote $\sigma_i^2 = \operatorname{Var} X_i$. Assume that there exist numbers $\sigma > 0$ and $\zeta > 0$ such that for all i, one has $\frac{1}{\sigma} < \sigma_i^2 < \sigma$ and $\mathbb{E}|X_i|^3 < \zeta$. In what follows, by "constant" we mean a quatity that may depend on σ and ζ, but not on anything else.

(1) Prove that for all i and all $t \in \mathbb{R}$, the first three derivatives of the characteristic function $\varphi_{X_i}(t)$ exist and are bounded from above by a constant. Prove that $|\varphi_{X_i}(t)| > \frac{1}{2}$ provided that $|t| < \varepsilon$, where ε is a constant.

(2) Prove that

$$\varphi_{X_i}(t) = e^{-\frac{t^2\sigma_i^2}{2} + \alpha(t)},$$

where $|\alpha(t)| \le C|t|^3$ provided that $|t| < \varepsilon$, and C and ε are constants.

(3) Denote $Y_n = \frac{1}{s_n}\sum_{i=1}^n X_i$, where $s_n = \sqrt{\sigma_1^2 + \dots + \sigma_n^2}$. Prove that for all $t \in \mathbb{R}$, we have

$$\varphi_{Y_n}(t) \overset{n\to\infty}{\longrightarrow} e^{-\frac{t^2}{2}}.$$

(4) Conclude that $Y_n \xrightarrow{\mathcal{D}} \mathcal{N}(0,1)$.

Exercise 5. (Non-symmetric stable distributions) Let $\alpha \in (0;2)$ and $\beta \in [-1;1]$. Let $X_1, X_2, \dots$ be i. i. d. random variables with density

$$f_{\alpha,\beta} = \begin{cases} \frac{1+\beta}{2}\alpha x^{-1-\alpha} & ,x \ge 1 \\ \frac{1-\beta}{2}\alpha |x|^{-1-\alpha}, & x \le -1 \\ 0, & x \in (-1;1) \end{cases}$$

(1) Prove that for all $t \neq 0$, one has

$$\varphi_{X_1}(t) = \alpha|t|^\alpha \int_{|t|}^\infty (\cos x + i\beta(\operatorname{sgn} t)\cdot \sin x) x^{-1-\alpha} dx.$$

(2) Prove that for $\alpha \neq 1$, we have, as $|t| \to 0$

$$\int_{|t|}^\infty x^{-1-\alpha}\sin x dx = \begin{cases} C_1 + o(1), & \alpha < 1 \\ C_2|t|^{1-\alpha} + C_3 + o(1), & \alpha > 1, \end{cases}$$

where $C_1, C_2, C_3 \in \mathbb{R}$ are constants that may depend on α (but not on t).

(3) Prove that, as $|t| \to 0$, one has

$$\log(\varphi_X(t)) = \begin{cases} C_4|t|^\alpha(1 + C_5 i\beta \mathrm{sgn} t) + o(|t|^\alpha), & \alpha < 1 \\ C_6 t + C_4|t|^\alpha(1 + C_5 i\beta \mathrm{sgn} t) + o(|t|^\alpha), & \alpha > 1, \end{cases}$$

where $C_4, \ldots, C_6 \in \mathbb{R}$ may depend on α (but not on β, t).

(4) Prove that if $X_1, X_2, \ldots$ are i. i. d. with density $f_{\alpha,\beta}$, then

$$n^{-\frac{1}{\alpha}} \left(\sum_{i=1}^{n} X_i - \mu n \right) \xrightarrow{\mathcal{D}} X,$$

where X is a random variable such that $\varphi_X(t) = e^{C_4|t|^\alpha(1+C_5 i\beta \mathrm{sgn} t)}$, and

$$\mu = \begin{cases} \mathbb{E}X_1, & \alpha > 1 \\ \text{any number}, & \alpha < 1. \end{cases}$$