Probability theory I - exercise set VI

Exercise 1. Prove that if $X_i \xrightarrow{\mathcal{D}} X$, then the sequence X_i is tight.

Exercise 2. Give examples of scalar random variables $X_1, X_2, \dots$ such that

- $X_i \xrightarrow{\mathcal{P}} 0$, but $X_i \overset{L^1}{\nrightarrow} 0$.
- $X_i \xrightarrow{L^1} 0$, but $X_i \overset{L^2}{\nrightarrow} 0$.
- $X_i \xrightarrow{\mathcal{P}} 0$, but almost surely, $X_i \nrightarrow 0$.

Exercise 3. Let $p \geq 1$. Prove that if $X_i \xrightarrow{\mathcal{P}} X$, and there exists a random variable $Y \geq 0$ with $\mathbb{E}Y^p < \infty$ such that for all i, $|X_i| \leq Y$ almost surely, then $X_i \xrightarrow{L^p} X$. Hint: you may find useful Minkowski's inequality

$$(\mathbb{E}|X+Y|^p)^{\frac{1}{p}} \leq (\mathbb{E}|X|^p)^{\frac{1}{p}} + (\mathbb{E}|Y|^p)^{\frac{1}{p}} .$$

Exercise 4. Assume that the events $E_1, E_2, \dots$ are independent, and denote $N(\omega) := \#\{i : \omega \in E_i\}$. Prove that if $\sum_{i=1}^{\infty} \mathbb{P}(E_i) = \infty$, then $\mathbb{P}(N = \infty) = 1$.

Exercise 5. Let $X_1, X_2, \dots$ be i. i. d., centered random variables and let $k \in \mathbb{N}$ be such that $\mathbb{E}X_1^{2k} < \infty$. Prove that there exists a constant $C > 0$ such that

$$\mathbb{P}\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i\right| > \varepsilon\right) \leq \frac{C}{\varepsilon^{2k} n^k}.$$

Exercise 6. Compute the characteristic functions of an exponential random variable and of $\frac{1}{2}$-Bernoulli random variable.

Exercise 7. Let F_X, F_Y be distribution functions of random variables X, Y. Define the Lévy's distance between F_X and F_Y by

$$d(F_X, F_Y) := \inf\{\delta > 0 : F_X(x-\delta) - \delta \leq F_Y(x) \leq F_X(x+\delta) + \delta \text{ for all } x \in \mathbb{R}\}.$$

Prove that $d(F_X, F_Y)$ is a metric on the set of all distribution functions, and that $d(F_{X_i}, F_X) \to 0$ if and only if $X_i \xrightarrow{\mathcal{D}} X$.