Osittaisdifferentiaaliyhtälöt
DEMO 8

1. Suppose that u is a smooth solution of the heat equation

$$u_t - \Delta u = 0$$

in $\mathbb{R}^n \times (0, \infty)$. Show that
i) $v(x,t) = u(\lambda x, \lambda^2 t)$ is a solution of the heat equation for each $\lambda \in \mathbb{R}$.
ii) $w(x,t) = x \cdot Du(x,t) + 2tu_t(x,t)$ is also a solution of the heat equation.

2. Let $u(x,t) = v(\frac{x^2}{t}), x \in \mathbb{R}, t \in (0, \infty)$.
i) Show that

$$u_t - u_{xx} = 0$$

if and only if

$$4zv''(z) + (2+z)v'(z) = 0. \tag{1}$$

ii) Show that

$$v(z) = C_1 \int_0^z e^{-s/4} s^{-1/2}\, ds + C_2$$

is a solution of equation (1).
iii) Differentiate $v(x^2/t)$ with respect to x to obtain the fundamental solution.

3. Solve the following initial-boundary value problem

$$\begin{cases} u_t - \Delta u = 0, & \text{in } B(0,1) \times (0, \infty); \\ u(x,0) = v(x), & \text{for } x \in B(0,1); \\ u(x,t) = 0, & \text{for } x \in \partial B(0,1), t > 0, \end{cases}$$

where v is an eigenfunction of $-\Delta$ in $B(0,1)$, that is,

$$-\Delta v = \lambda v, \text{ in } B(0,1)$$

for an eigenvalue λ.