Osittaisdifferentiaaliyhtälöt
DEMO7

1. Let u be a smooth solution of

$$\begin{cases} -\Delta u = f & \text{in } B(0,1) \subset \mathbb{R}^n; \\ u = g & \text{on } \partial B(0,1). \end{cases}$$

Prove that

$$\max_{B(0,1)} |u| \leq c \left(\max_{\partial B(0,1)} |g| + \max_{B(0,1)} |f| \right),$$

where $c = c(n) > 0$.

2. Use Poisson's formula to prove that

$$r^{n-2} \frac{r - |x|}{(r + |x|)^{n-1}} u(0) \leq u(x) \leq r^{n-2} \frac{r + |x|}{(r - |x|)^{n-1}} u(0),$$

whenever u is positive and harmonic in $B(0,r) \subset \mathbb{R}^n$.

3. Prove that

$$\pi^2 = \inf \left\{ \int_0^1 (w')^2 \, dx : w \in C^2(0,1) \cap C^1([0,1]), w(0) = w(1) = 0, \int_0^1 w^2 \, dx = 1 \right\}.$$

4. Calculate the principal eigenvalue $\lambda_1(\Omega)$ of $-\Delta$ for

$$\Omega = \{(x,y) \in \mathbb{R}^2 : 0 < x < 1, 0 < y < 1\}.$$

(Hint: consider $u(x,y) = \sin(\pi x) \sin(\pi y)$.)

5. Let Ω be a bounded smooth domain in $\mathbb{R}^n$. Suppose that the following Neumann problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega; \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

has a non-trivial solution $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$. Show that $\lambda \geq 0$.