Osittaisdifferentiaaliyhtälöt
DEMO 4

1. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Suppose that $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies

$$-\Delta u \leq 0 \quad \text{in } \Omega.$$

Prove that

i)

$$u(x) \leq \fint_{B(x,r)} u(y)\, dy \quad \text{for all balls } B(x,r) \subset \Omega;$$

ii)

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

2. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a smooth and convex function. Assume that u is harmonic in Ω. Let $v = \varphi(u)$. Prove that

$$-\Delta v \leq 0 \quad \text{in } \Omega.$$

3. Assume that u is harmonic in Ω. Let $v = |Du|^2$. Prove that

$$-\Delta v \leq 0 \quad \text{in } \Omega.$$

4. Let $B(0,r)$ be a ball in $\mathbb{R}^n, n \geq 3$. Suppose that $u \in C^2(B(0,r)) \cap C(\overline{B(0,r)})$ is a solution to the boundary value problem

$$\begin{cases} -\Delta u = f & \text{in } B(0,r); \\ u = g & \text{on } \partial B(0,r). \end{cases}$$

Modify the proof of the mean value formula to show that

$$u(0) = \fint_{\partial B(0,r)} g(x)\, dS(x) + \frac{1}{n(n-2)\alpha_n} \int_{B(0,r)} \Big(\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}} \Big) f(x)\, dx.$$

5. Prove that there is a unique solution $u \in C^2(B(0,1)) \cap C(\overline{B(0,1)})$ to the following boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in } B(0,1); \\ u = 1 & \text{on } \partial B(0,1). \end{cases}$$

6. Prove that there is a unique solution $u \in C^2(B(0,1)) \cap C(\overline{B(0,1)})$ to the following boundary value problem

$$\begin{cases} \Delta u = u^3 & \text{in } B(0,1); \\ u = 0 & \text{on } \partial B(0,1). \end{cases}$$

(Hint: apply conclusion ii) of problem 1 in this exercise.)