Osittaisdifferentiaaliyhtälöt
DEMO 3

1. Write down an explicit formula for a function u solving

$$\begin{cases} u_t + b \cdot Du + cu = 0 & \text{in } \mathbb{R}^n \times (0, \infty); \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases} \tag{1}$$

Here $c \in \mathbb{R}$ and $b \in \mathbb{R}^n$ are constants.

Solution. We define a new function $v : \mathbb{R}^n \times (0, \infty) \to \mathbb{R}$ as follows

$$v(x, t) = u(x, t) \exp(ct).$$

Then it is easy to check that function v is a solution to the following initial value problem

$$\begin{cases} v_t + b \cdot Dv = 0 & \text{in } \mathbb{R}^n \times (0, \infty); \\ v = g & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

The unique solution to the above linear transport equation is

$$v(x, t) = g(x - tb).$$

Thus the unique solution to the initial value problem (1) is

$$u(x, t) = g(x - tb) \exp(-ct).$$

□

2. Let $u \in C^1(\mathbb{R}^n)$ and

$$\varphi(r) = ⨍_{\partial B(x,r)} u(y)\, dS(y).$$

Prove that

$$\varphi'(r) = ⨍_{\partial B(0,1)} Du(x + ry) \cdot y\, dS(y).$$

Proof. We rewrite φ by changing variables as follows

$$\varphi(r) = ⨍_{\partial B(0,1)} u(x + ry)\, dS(y).$$

Now fix $r > 0$, and let $h \in \mathbb{R}$. We consider the difference quotient

$$\frac{\varphi(r + h) - \varphi(r)}{h} = ⨍_{\partial B(0,1)} \frac{u(x + ry + hy) - u(x + ry)}{h}\, dS(y). \tag{2}$$

Now for the integrand function of the integral on the right hand side, we have by the fundamental theorem of calculus that

$$\begin{aligned} \frac{u(x + ry + hy) - u(x + ry)}{h} &= \frac{1}{h} \int_0^1 \frac{d}{dt} u(x + ry + thy)\, dt \\ &= \int_0^1 Du(x + ry + thy) \cdot y\, dt, \end{aligned} \tag{3}$$

where the second equality follows from the chain rule and the assumption that $u \in C^1(\mathbb{R}^n)$. It follows from (2) and (3) that

$$\begin{aligned} &\frac{\varphi(r+h)-\varphi(r)}{h} - \fint_{\partial B(0,1)} Du(x+ry)\cdot y\, dS(y) \\ (4)\quad &= \fint_{\partial B(0,1)} \int_0^1 \big(Du(x+ry+thy) - Du(x+ry)\big)\cdot y\, dt\, dS(y). \end{aligned}$$

Now since $u \in C^1(\mathbb{R}^n)$, we have that $\partial_{x_i} u$ is continuous in $\mathbb{R}^n$ for each $i = 1, 2, ..., n$. Thus it is uniformly continuous in $\overline{B(x, r+|h|)}$. This means that for any $\varepsilon > 0$, there is $\delta > 0$, depending only on ε, such that

$$(5)\qquad |\partial_i u(z) - \partial_i u(w)| < \varepsilon / n^{1/2}$$

for each $i = 1, 2, ..., n$ and for all $z, w \in \overline{B(x, r+|h|)}$ such that

$$|z - w| < \delta.$$

It follows from (5) that

$$(6)\qquad |Du(z) - Du(w)| < \varepsilon$$

for all $z, w \in \overline{B(x, r+|h|)}$ such that

$$|z - w| < \delta.$$

Now we go back to (4). Note that

$$x + ry + thy \in \overline{B(x, r+|h|)}, \quad x + ry \in \overline{B(x, r+|h|)}$$

for all $y \in \partial B(0,1)$ and all $t \in [0,1]$. Note also that

$$|x + ry + thy - (x + ry)| = t|h|.$$

Thus if $|h| < \delta$, we have by (6) that

$$(7)\qquad |Du(x+ry+thy) - Du(x+ry)| < \varepsilon$$

for all $y \in \partial B(0,1)$ and all $t \in [0,1]$. It follows easily from (4) and (7) that

$$\Big|\frac{\varphi(r+h)-\varphi(r)}{h} - \fint_{\partial B(0,1)} Du(x+ry)\cdot y\, dS(y)\Big| < \varepsilon.$$

The above inequality holds, provided that $|h| < \delta$. This show that

$$\lim_{h\to 0} \frac{\varphi(r+h)-\varphi(r)}{h} = \fint_{\partial B(0,1)} Du(x+ry)\cdot y\, dS(y),$$

from which it follows that

$$\varphi'(r) = \fint_{\partial B(0,1)} Du(x+ry)\cdot y\, dS(y).$$

The proof is complete. □

3. Find a solution to the following boundary value problem

$$\begin{cases} \Delta u = u^3 & \text{in } B(0,1); \\ u = 0 & \text{on } \partial B(0,1). \end{cases}$$

Solution. It is obvious that

$$u(x) = 0, \quad x \in B(0,1),$$

is a solution. □

4. Find a solution to the following boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in } B(0,1); \\ u = 1 & \text{on } \partial B(0,1). \end{cases}$$

Solution. It is obvious that

$$u(x) = 1, \quad x \in B(0,1),$$

is a solution. □

5. Find a solution to the following boundary value problem

$$\begin{cases} \Delta u = 1 & \text{in } \Omega = \{x \in \mathbb{R}^3 : a < |x| < b\}; \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{8}$$

where $0 < a < b < \infty$.

Solution. We will find a solution of the following form

$$u(x) = c_1|x|^{-1} + c_2|x|^2 + c_3,$$

where c_1, c_2, c_3 are constants. An easy calculation shows that

$$\Delta u(x) = 6c_2$$

for $x \neq 0$. Therefore if we set $c_2 = 1/6$, then u is a solution to equation

$$\Delta u = 1 \quad \text{in } \Omega.$$

In order to satisfy the boundary condition $u = 0$ on $\partial\Omega$, we need

$$\begin{cases} c_1 a^{-1} + c_2 a^2 + c_3 = 0; \\ c_1 b^{-1} + c_2 b^2 + c_3 = 0. \end{cases}$$

From the above system, we obtain that

$$c_1 = c_2 ab(a+b), \quad c_3 = -c_2(a^2 + ab + b^2).$$

Thus

$$u(x) = \frac{1}{6}\left[ab(a+b)|x|^{-1} + |x|^2 - (a^2 + ab + b^2)\right]$$

is a solution to Dirichlet problem (8). □

6. Prove that Laplace's equation

$$\Delta u = 0$$

is rotation invariant; that is, if O is an orthogonal $n \times n$ matrix and we define

$$v(x) = u(Ox),$$

then

$$\Delta v = 0. \tag{9}$$

Proof. Let $u = u(y)$ be a harmonic function, that is, $u \in C^2$ is a solution of Laplace's equation

$$\Delta u = 0. \tag{10}$$

Now fix an orthogonal matrix $O = (o_{ij})$ and let $f = (f^1, f^2, ..., f^n)$ be the linear map

$$f(x) = Ox,$$

that is,

$$f^i(x) = \sum_{k=1}^{n} o_{ik} x_k$$

for all $i = 1, 2, ..., n$. We note that

$$\partial_{x_j} f^i(x) = o_{ij} \tag{11}$$

for all $i, j = 1, 2, ..., n$.

Now let $v(x) = u(f(x))$. By chain rule, we have for $i = 1, 2, ..., n$ that

$$\partial_{x_i} v(x) = \sum_{k=1}^{n} \partial_{y_k} u(f(x)) \partial_{x_i} f^k(x) = \sum_{k=1}^{n} o_{ki} \partial_{y_k} u(f(x)),$$

and that

$$\partial_{x_i x_i} v(x) = \sum_{k,l=1}^{n} o_{ki} \partial_{y_l y_k} u(f(x)) \partial_{x_i} f^l(x) = \sum_{k,l=1}^{n} o_{ki} o_{li} \partial_{y_l y_k} u(f(x)).$$

The above equality hold for all $i = 1, 2, ..., n$. Thus we have that

$$\Delta v(x) = \sum_{k,l=1}^{n} \big(\sum_{i=1}^{n} o_{ki} o_{li} \big) \cdot \partial_{y_l y_k} u(f(x)). \tag{12}$$

Since O is an orthogonal matrix, we have that

$$O^T O = I,$$

which means that

$$\sum_{i=1}^{n} o_{ki} o_{li} = \delta_{kl}$$

for all $k, l = 1, 2, ..., n$, where $\delta_{kl} = 1$ if $k = l$ and $\delta_{kl} = 0$ if $k \neq l$. Thus (12) becomes

$$\Delta v(x) = \sum_{k,l=1}^{n} \delta_{kl} \partial_{y_l y_k} u(f(x)) = \sum_{k=1}^{n} \partial_{y_k y_k} u(f(x)) = \Delta u(f(x)),$$

which, together with (10), gives the conclusion (9). □