Osittaisdifferentiaaliyhtälöt
DEMO 10

1. Find a solution to the following Cauchy problem

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = xt & \text{in } \mathbb{R} \times (0,\infty); \\ u(x,0) = x & \text{on } \mathbb{R} \times \{t=0\}. \end{cases}$$

2. Let u be a smooth solution of the following initial-boundary problem

$$\begin{cases} u_t - \Delta u = u & \text{in } \Omega_T = \Omega \times (0,T); \\ u = g & \text{on } \Gamma_T = (\Omega \times \{t=0\}) \cup (\partial\Omega \times [0,T]), \end{cases}$$

where Ω is a bounded smooth domain, $T > 0$, and g is a continuous function. Prove that

$$|u(x,t)| \le e^t \max_{\Gamma_T} |g|, \quad \forall\, (x,t) \in \Omega_T.$$

3. Prove that

$$u(x,t) = \frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{(t-s)^{3/2}} e^{-\frac{x^2}{4(t-s)}} h(s)\, ds$$

is a solution of the PDE

$$\begin{cases} u_t - u_{xx} = 0, & x > 0, t > 0; \\ u(x,0) = 0, & x > 0; \\ u(0,t) = h(t), & t > 0, \end{cases}$$

where h is a smooth bounded function with $h(0) = 0$.

4. Find a solution to the inhomogeneous diffusion equation on the half-line with Dirichlet boundary condition:

$$\begin{cases} u_t - u_{xx} = f(x,t), & x > 0, t > 0; \\ u(x,0) = \varphi(x), & x > 0; \\ u(0,t) = 0, & t > 0 \end{cases}$$

where φ is a smooth bounded function with $\varphi(0) = 0$.