UH Malliavin Calculus, Fall 2016, Exercises 7 (30.11, 7.12 2016)

1. Consider the Black and Scholes stochastic differential equation

$$dS_t = S_t(b(S_t)dt + a(S_t)dW_t)$$

and assume that b and a are C^1 functions. Compute the Malliavin derivative of $D_u S_t$ and Write the stochastic differential equation for the Malliavin derivative $D_u S_t$, where u is fixed and $t \in [0, T]$.

2. Let ξ_i, $i \leq n$ i.i.d. standard Gaussian random variables, and let A be a $n \times n$ matrix with $\det(A) \neq 0$.

Consider the random variable

$$F_i = \sum_{j=1}^{n} A_{ij}(\xi_j^2 - 1) = \sum_{j=1}^{n} A_{ij} h_2(\xi_j), \quad , i = 1, \dots, n.$$

where $h_2(x)$ is the 2nd Hermite polynomial.

(a) Write the random $n \times n$ matrix Γ with

$$\Gamma_{i,j} = \langle DF_i, DF_j \rangle_H$$

where you can take as Hilbert space $H = \mathbb{R}^n$ or

$$H = \ell^2 = \Big\{ \ a = (a_i : i \in \mathbb{N}) : \| \ a \ \|_{\ell^2}^2 = \sum_i a_i^2 < \infty \Big\}$$

(b) Prove that Γ is invertible.

(c) Use the integration by part formula of Malliavin calculus to show that:

for smooth $g : \mathbb{R}^n \to \mathbb{R}$ and denoting $\partial_i = \frac{\partial}{\partial x_i}$, by

$$E\big(Y \partial_i g(F)\big) = E\big(g(F)\Psi_i^1(Y, F)\big)$$

where

$$\Psi_i^1(Y, F) = \delta\bigg(Y \sum_{j=1}^{n} \Gamma_{i,j}^{-1} DF_j\bigg)$$

see Baudoin book page 216, see also Nualart, Proposition 2.1.4. p 100 in Nualart book (we have done this in the lectures for Y=1). and by induction show

$$E\big(Y \partial_{i_1,\dots,i_m}^m g(F)\big) = E\big(g(F)\Psi_{i_1,\dots,i_m}^m(Y, F)\big)$$

where

$$\Psi^m_{i_1,\dots,i_m}(Y,F) = \delta\left(\Psi^{m-1}_{i_1,\dots,i_{m-1}}(Y,F)\sum_{j=1}^{n}\Gamma^{-1}_{i,j}DF_j\right)$$

(d) Apply this formula to compute an expession for the joint density of the vector $F = (F_1, \dots, F_n)$ with respect to the Lebesgue measure in $\mathbb{R}^n$.

$$p_F(x_1,\dots,x_n) = \frac{\partial^n}{\partial_{x_1},\dots,\partial_{x_n}}P\big(X_1 \le x_1,\dots,X_n \le x_n\big)$$
$$= \frac{\partial^n}{\partial_{x_1},\dots,\partial_{x_n}}E\big(\mathbf{1}(F_1 \le x_1)\times\cdots\times\mathbf{1}(F_n \le x_n)\big)$$

To keep the calculation short you can assume that $n = 2$.

(e) Use the result to plan a Monte Carlo experiment computing an approximation of the density $p_F(x)$.