UH Malliavin Calculus, Fall 2016, Exercises 6 (16 November 2016)

1. (a) Let $f_1 \neq \cdots \neq f_n \in H$. Find $m = m(n)$, coefficents $c_i \in \mathbb{R}$ and $g_1, \ldots, g_m \in H$ such that

$$f_1 \widetilde{\otimes} f_2 \widetilde{\otimes} \ldots \widetilde{\otimes} f_n = \sum_{i=1}^{m} c_i g_i^{\otimes n} \in H^{\widetilde{\otimes} n}$$

   (b) Write the iterated integral $I_n(f_1 \otimes f_2 \otimes \cdots \otimes f_n)$ as linear combination of $h_n\big(X(\varphi_i)\big)$ where $\varphi_i \in H$ and $\| \varphi_i \|_H = 1$, and $h_n = \partial^{*n} 1$ is the unnormalized n-th Hermite polynomial.

   Hint: when $n = 2$, $f_1, f_2 \in H$ we have the polarization identity

$$4 \; f_1 \widetilde{\otimes} f_2 = (f_1 + f_2) \otimes (f_1 + f_2) - (f_1 - f_2) \otimes (f_1 - f_2)$$

   so that

   $4 \; I_2(f_1 \otimes f_2) =$

   $\| f_1 + f_2 \|_H^n \; h_n\big(X(f_1 + f_2)/ \| f_1 + f_2 \|_H\big) - \| f_1 - f_2 \|_H^n \; h_n\big(X(f_1 - f_2)/ \| f_1 - f_2 \|_H\big)$

   You can try first with $n = 3$ before looking at the general case.

2. We recall Stroock formula: if $F \in D^{n,2}$ for all n, i.e. F has infinitely many Malliavin differentiable infinitely many times in L^2, it is possible to write the chaos expansion as

$$F(\omega) = E(F) + \sum_{n=1}^{\infty} I_n(f_n) = E(F) + \sum_{n=1}^{\infty} \frac{1}{n!} I_n(E(D^n F))$$

   Note that $D^n_{t_1,\ldots,t_n} F(\omega)$ is a symmetric random function of n arguments. This can be useful to compute chaos expansions.

   Notation : We denote by $\delta(u) = \int u_s \delta W_s$ the Skorokhod integral while $\int u_s dW_s$ is the Ito integral when u_s is adapted to the Brownian filtration.

   Compute the following divergence (Skorohod) integrals:

$$i) \qquad \delta(u) = \int_0^T W_T^2 W_s^2 \delta W_s$$

$$ii) \qquad \delta(u) = \int_0^T \exp(W_T) W_s^2 \delta W_s$$

$$iii) \qquad \delta(u) = \int_0^T \exp(W_T - W_t) W_t \delta W_t$$

Hint: use Stroock formula to compute the chaos expansion of the integrand, and then use the linearity of δ. Also recall that

$$\delta(Fu) = F\delta u - \langle u, DF\rangle_H$$

when $F \in \mathbb{D}^{1,2}$ and $u \in L^2(\Omega; H)$.

3. Compute the Malliavin derivatives $D_t(\delta(u))$ for the divergence integrands in 2.i), 2.ii), 2.iii)
   We can either take directly the Malliavin derivative, or use the formula

   $$D\delta(u) = u + \delta(D_u)$$

   which means

   $$D_t(\delta(u)) = u_t + \int_0^T D_t u_s \delta W_s$$

   ii) $F(\omega) = \sin(W_T)$, iv) $F(\omega) = (W_T + T)\exp(-W(T) - \frac{1}{2}T)$
   Note that $Z_t := \exp(-W(t) - \frac{1}{2}t) = \frac{dQ_t}{dP_t}$ is the Radon-Nykodim derivative which appears in Girsanov theorem. Under the measure Q the process $W_t := W_t + t$ is a standard Brownian motion, while under the measure P it is a Brownian motion with drift coefficient 1.

4. (a) Let $A \in \mathcal{F}^X = \sigma(X(h) : h \in H)$, and $F(\omega) = \mathbf{1}_A(\omega)$.
   Show that F is Malliavin differentiable if and only if $P(A) = 0$ or $P(A) = 1$.
   Hint: Assume that the Malliavin derivative DF exists, take Malliavin derivative of both sides in the identity $F = F^2$ and find a contradiction.

   (b) Find the Ito-Clarck representation of the digital option

   $$F = 1(W_T > 0),$$

   where W_t is Brownian motion.
   Hint: Although we don't have enough smoothness of the functional, use formally the Clarck Ocone formula, where we compute the Malliavin derivative by chain rule obtaining a Dirac delta function, write the conditional expectation explicitely and use the gaussian integration by parts formula, and show that the final result still makes sense (this is the hardest part).

Exercise 6 The local time process L_t^0 of the Brownian motion at 0 is defined as

$$L_t^0 = |W_t| - \int\limits_0^t \mathrm{sign}(W_s) dW_s$$

where $\mathrm{sign}(x) = 2\,\mathbf{1}(x \geq 0) - 1$. The local time L_t^0 is interpreted as the time the Brownian motion W_t spends at the value 0 within the time interval $[0, t]$.

(a) Is L_t^0 Malliavin differentiable ?

(b) Find the Ito Clarck representation of L_T^0. Hint: as in Exercise 5, compute formally the Malliavin derivative, take conditional expection and use Ito Clarck Ocone formula. Explain why the obtained formula is correct.