UH Malliavin Calculus, Fall 2016, Exercises 5 (19 and 26 October 2016)

On the Wiener space, where $H = L^2([0,T], \mathcal{B}, dt)$ and $W_t = W(1_{[0,t]})$ $0 \le t \le T$ is Brownian motion. Let $\mathcal{F}_A = \sigma(W(h\mathbf{1}_A) : h \in H)$ and $\mathcal{F}_t = \mathcal{F}_{[0,t]}$, $t \in [0,T]$.

1. Consider the Black-Scholes process

$$S_t = S_0 \exp\bigl(\sigma W_t + (r - \sigma^2/2)t\bigr)$$

Under P the discounted process $\widetilde{S}_t = S_t e^{-rt}$ is an exponential martingale, and the asian option

$$F(\omega) = \left(\frac{1}{T}\int_0^T S(t)dt - K\right)^+$$

which pays at maturity T the difference between the average stock price and the strike price K when this difference is positive

Use the Ito Clarck Ocone formula to find the martingale representation of F with respect to the Brownian motion W_t, and use the representation

$$dW_t = \frac{dS_t}{S_t} - r dt$$

to find the hedging strategy.

2. Let $f(t_1, t_2, t_3) = t_1^2 t_2 t_3$, $t_i \in [0,T]$.

   (a) Write the symmetrization $\widetilde{f}(t_1, t_2, t_3)$.

   (b) Write $I_3(f)$ as iterated Ito integral in the interval $[0,T]$.

   (c) Write its Malliavin derivative $D_t I_3(f)$.

   (d) Write its second Malliavin derivative $D^2_{s,t} I_3(f)$

   (e) Let $u(t) = I_3(f)\sin(t)$. Is $u(t)$ adapted ?

   (f) Write the Skorokhod integral $\delta(u)$ over $[0,T]$.

   (g) Write the Ito Clarck Ocone martingale representation of $\delta(u)$.

3. Let $h_n(x) = (\partial^{*n} 1)(x)$ be the unnormalized Hermite polynomial. Remember that

$$\frac{d}{dx} h_n(x) = n h_{n-1}(x) \qquad \text{and } P_t h_n(x) = e^{-nt} h_n(x)$$

where $(P_t : t \geq 0)$ is the Ornstein Uhlenbeck semigroup on $L^2(\mathbb{R}, \mathcal{B}(\mathbb{R}), \gamma)$. Let also be $f \in H = L^2([0,T], dt)$ deterministic and consider the random variable $F = h_n(W(f)) = h_n\Big(\int_0^T f(s)dW(s) \Big)$.

(a) Use the OU-semigroup to compute for $0 \leq t \leq T$

$$E_P(h_n(W(f))|\mathcal{F}_t^W) \ .$$

Hint: when you compute the conditional expectation use the representation

$$W(f) = W(f\mathbf{1}_{[0,t]}) + W(f\mathbf{1}_{[t,T]})$$

where the first Gaussian r.v. on the right is $\mathcal{F}_t^W$ and the second Gaussian r.v. is independent from $\mathcal{F}_t^W$.

(b) Show by explicit computation that

$$D_u E_P(h_n(W(f))|\mathcal{F}_t^W) = E_P(D_u h_n(W(f))|\mathcal{F}_t^W)\mathbf{1}(u \leq t) \quad (0.1)$$

Define the *optional projection* of the Malliavin derivative $D_t F$ process (which is not necessarly $\mathbb{F}^W$-adapted) as the $\mathbb{F}^W$-adapted) as the $\mathbb{F}^W$-adapted process obrained by taking the $\mathcal{F}_t$-conditional expectation of $D_t F$ at every time $t \in [0,T]$:

$$^o D_t F = E_P(D_t h_n(W(f))|\mathcal{F}_t^W) = D_t E_P(h_n(W(f))|\mathcal{F}_t^W)$$

where we just plug-in $u = t$ in (0.1).

For a constant random variable F we define its optional projection as the $\mathbb{F}^W$ adapted martingale $\left(^o F\right)_t = E(F|\mathcal{F}_t)$.

(c) Recall that for $f \in H = L^2([0,T], dt)$ $F = W(f) = \int_0^T f(s)dW_s$ where on the right side we have a Wiener Ito integral. For $t \in [0,T]$, we can interpret the Wiener-Ito integral

$$W(f\mathbf{1}_{[0,t]}) = \int_0^t f(s)dW_s = E(W(f)|\mathcal{F}_t)$$

as a process indexed by t, which is a Gaussian martingale in the filtration $\mathbb{F}^W$.

We also recall Ito formula: for $\varphi(x,t) \in C^{2,1}$ (twice differentiable w.r.t. x and differentiable w.r.t. t) and $X_t = X_0 + \int_0^t Y_s dW_s$ Ito integral with $Y_s(\omega) \in L^2(\Omega \times [0,T], dP \times ds)$ and adapted

$$\varphi\big(X_t, t\big) = \varphi(X_0, 0) + \int_0^t \frac{\partial \varphi}{\partial x}\big(X_s, s\big) Y_s dW_s +$$
$$\frac{1}{2}\int_0^t \frac{\partial^2 \varphi}{\partial x^2}\big(X_s, s\big) Y_s^2 ds + \int_0^t \frac{\partial \varphi}{\partial s}\big(X_s, s\big) ds$$

Compute the Ito differential of

$$W_t/\sqrt{t} \tag{0.2}$$

and the Ito differential of

$$W(f\mathbf{1}_{[0,t]})/ \parallel f\mathbf{1}_{[0,t]} \parallel_H = \int_0^t f(s) dW_s \Big/ \sqrt{\int_0^t f(s)^2 ds}$$

(d) Apply Ito formula and the properties of the Hermite polynomials to represent

$$t^{n/2} h_n\big(W_t/\sqrt{t}\big)$$

as Ito integral.

(e) Do the same for

$$\parallel f\mathbf{1}_{[0,t]} \parallel_H^n h_n\big(W(f\mathbf{1}_{[0,t]})/ \parallel f\mathbf{1}_{[0,t]} \parallel_H\big) =$$
$$\Big(\int_0^t f(s)^2 ds\Big)^{n/2} h_n\Big(\int_0^t f(s) dW_s \Big/ \sqrt{\int_0^t f(s)^2 ds}\Big)$$

(f) Show that when $\parallel f \parallel_H = 1$ this representation coincides with the Clarck Ocone formula for $F(\omega) = h_n(W(f))$

$$F = E(F) + \int_0^T E(D_t F | \mathcal{F}_s^W) dW_s$$

Note that this proves the Ito Clarck Ocone formula for $F(\omega) = h_n(W(f))$

(g) Use now the definition of Skorokhod integral and the fact the Ito and Skorokhod integral coincide for adapted integrands to show that for $F \in D^{1,2}$ the Clarck Ocone formula holds in general

$$F - E(F) = \delta({}^oDF) = \int_0^T {}^oD_t F dW_t \; .$$

Hint. Remember that linear combinations of random variables of the form $h_n(W(f))$ with $\parallel f \parallel_H = 1$ are dense in $L^2(\Omega, \mathcal{F}, P)$.

4. Show that

$$E\big((F-E(F))^2\big) = E\big(\langle {}^oDF, DF\rangle_H\big) = E\big(\langle {}^oDF, {}^oDF\rangle_H\big) \le E\big(\langle DF, DF\rangle\big)$$

Show that the operator $F : D^{1,2} \to {}^o(DF) = E(DF|\mathcal{F}_t) = DE(F|\mathcal{F}_t) = D^oF$ is closed, meaning that if $F_n \in D^{1,2}$ and $F_n \stackrel{L^2(\Omega;\mathbb{R})}{\longrightarrow} 0$ and ${}^oDF_n \stackrel{L^2(\Omega;H)}{\longrightarrow} \eta$ then $\eta = 0$.

(b) Show that ${}^oDF$ can be extended from $D^{1,2}$ to all F, $L^2(\Omega, \mathbb{R})$, meaning that D^oF with $D_tE(F|\mathcal{F}_t)$ is a well defined adapted process for all $F \in L^2$, and the Ito Clark Ocone formula generalizes as

$$F - E(F) = \int_0^t D_t\big({}^oF\big)_t dW_t = \int_0^t D_tE(F|\mathcal{F}_t)dW_t$$

For any $F \in L^2(\Omega)$, (also when $F \notin \mathbb{D}^{1,2}$), we can always compute $D_tE(F|\mathcal{F}_t)$ as the limit in $L^2(\Omega; H)$ of $E(D_tF_n|\mathcal{F}_t)$ for a smooth approximating sequence $F_n \in D^{1,2}$ such that $F_n \stackrel{L^2(\Omega,R)}{\longrightarrow} F$.

5. Consider the symmetric functions $f(t_1, t_2) = t_1t_2^2 + t_1^2t_2$, $g(t_1, t_2, t_3) = t_1t_2t_3$.

   (a) compute $(f \otimes g)(t_1, t_2, t_3, t_4, t_5)$
   (b) compute its symmetrization $(f\widetilde{\otimes}g)(t_1, t_2, t_3, t_4, t_5)$
   (c) compute the contraction $(f \otimes_1 g)(t_1, t_2, t_3)$
   (d) compute the symmetrized contraction $(f\widetilde{\otimes_1}g)(t_1, t_2, t_3)$
   (e) Write down the iterated integral $I_3(f \otimes_1 g)$
   (f) Compute $D_tI_3(f \otimes_1 g)$
   (g) Compute the second Malliavin derivative $D^2_{s,t}I_3(f \otimes_1 g)$.