UH Malliavin Calculus, Fall 2016, Exercises 4 (5,12 October 2016)

1. Show that the Malliavin derivative defines for simple random variable

$$F = f(X(h_1), \dots, X(h_n)) \in \mathcal{S}$$

as

$$DF = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f(X(h_1), \dots, X(h_n)) h_i \in L^2(\Omega, H)$$

does not depend on the representation of F,

Note that the representation of $F \in \mathcal{S}$ is not unique, it could be written as

$$F = \varphi(X(\eta_1), \dots, X(\eta_m))$$

for some other m η_i and φ.

2. Let $\{\xi_n : n \in \mathbb{N}\}$ be a sequence of Gaussian random variables, $\xi_n \sim \mathcal{N}(\mu_n, \sigma_n)$ such that $\xi_n \overset{law}{\to} \xi$ with respect to the convergence in distribution, meaning that the characteristic function converges pointwise

$$\varphi_{\xi_n}(t) := E\big(\exp(it\xi_n)\big) = \exp\big(it\mu_n - t^2\sigma_n^2/2\big) \longrightarrow \varphi_{\xi}(t) = E\big(\exp(it\xi\big) \quad \forall t$$

where $i = \sqrt{(-1)}$ is the imaginary unit. Show that ξ is either Gaussian or deterministic.

Hint: show that μ_n and σ_n have limits.

3. For $g \in H$ we consider the shifted process $(X+g)(h) = X(h) + \langle g, h\rangle_H$, $\forall h \in H$, which are jointly Gaussian, with mean $\langle g, h\rangle_H$ and covariance $E((X+g)(h)(X+g)(h')) = E(X(h)X(h')) = \langle h, h\prime\rangle_H$.

We consider the shift operator $T_g : \mathcal{S} \longrightarrow \mathcal{S}$ such that

$$F = f\big(X(h_1), \dots, X(h_n)\big) \mapsto T_g F = f\big(X(h_1) + \langle g, h_1\rangle_H, \dots, X(h_n) + \langle g, h_n\rangle_H\big)$$

Show there is a probability measure $P_g \sim P$ (the shifted Gaussian measure) such that if $F \in \mathcal{S}$

$$E_P(T_g F) = E_{P_g}(F) = E_P(FZ), \quad Z = \frac{dP_g}{dP} = \exp\big(X(g) - \langle g, g\rangle_H/2\big)$$

Hint. Show that $Z > 0$ P a.s. and it is a Radon Nikodym derivative with $E_P(Z) = 1$.

4. Let $\{ e_i : i \in \mathbb{N}\}$ a complete orthonormal system of the separable Hilbert space, $\{ \xi_i(\omega) : i \in \mathbb{N}\}$ a sequence of i.i.d. standard Gaussian variables defines on $(\Omega, \mathcal{F}, P)$, and let

$$X(h) = \sum_{i=1}^{\infty} \langle h, e_i \rangle_H \xi_i(\omega)$$

be the isonormal Gaussian process. Show that the shifted isonormal process has the representation

$$(X+g)(h) \sum_{i=1}^{\infty} \Big(\langle h, e_i \rangle_H \xi_i(\omega) + \langle h, g \rangle_H \Big)$$

as a limit in $L^2(\Omega, \mathbb{R})$

5. Show that T_g is closed in $L^2(\Omega, \mathcal{F}, P)$, and it has an unique extension $T_g : \mathrm{Dom}(T_g) \to L^2(\Omega)$.

   We consider now a fix direction g, show for $F \in \mathcal{S}$ the *directional derivative* exists

$$D_g F := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}(T_{\varepsilon g} F - F) = \langle DF, g \rangle$$

   where we take the limit in $L^2(\Omega, \mathbb{R})$. Show that D_g is a closed operator which is extended uniquely to an operator defined on a closed subspace $\mathrm{Dom}(D_g) \supseteq D^{1,2}$.

6. Compute the divergence integrals with respect to the Brownian motion W, where $H = L^2([0,T], dt)$, and the isonormal Gaussian process is

$$W(h) = \int_0^t h(s) W(ds)$$

   is the Wiener integral

   (a) $\int W(t) \delta W(t)$

   (b) $\int W(t)^2 \delta W(t)$

   (c) $\int W(t_0)^2 \delta W(t)$ with fixed t_0

   (d) $\int_0^T \exp(W(T)) \delta W(t)$