UH Malliavin Calculus, Fall 2016, Exercises 3 (28.9 2016)

Let $N(\omega) \sim \mathcal{N}(0,1)$ is a standard Gaussian random variable, with probability density

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \exp\Big(-\frac{x^2}{2}\Big), \quad x \in \mathbb{R}$$

and $\gamma(dx) = \phi(x)dx$ is the standard Gaussian distribution on $\mathbb{R}$.

1. We denote by $H_n(x) = \big(\partial^{*n} 1\big)(x)$ the n-th unnormalized Hermite polynomial. Show that

$$H_p(x)H_q(x) = \sum_{r=0}^{\min\{p,q\}} r! \binom{p}{r}\binom{q}{r} H_{p+q-2r}(x)$$

   Hint: we have seen that for $f \in L^2(\gamma)$ $\{\ h_r(x) = H_r(x)/\sqrt{r!} :\ r \in \mathbb{N}\}$ is a complete orthonormal system in $L^2(\gamma)$ and for $f \in L^2(\gamma)$

$$f(x) = \sum_{r=0}^{\infty} \frac{E(f(x)H_r(x))}{r!} H_r(x)$$

   where the series is convergent in $L^2(\gamma)$.

   Use the expansion with $f(x) = H_p(x)H_q(x)$.

2. Show *Rodrigues formula*

$$H_p(x) = e^{x^2/2}(-1)^p \frac{d^p}{dx^p} e^{-x^2/2}$$

3. Show that

$$g(x,t) := \exp\big(tx - t^2/2\big) = \sum_{r=0}^{\infty} \frac{t^r}{r!} H_r(x)$$

   $g(x,t)$ is the *generating function* of the Hermite polynomials. Hint: $g(x,t) = \exp(x^2)\exp(-(t-x)^2/2)$

4. Show that

$$\exp\big(tx - t^2\lambda^2/2\big) = \sum_{r=0}^{\infty} \frac{t^2\lambda^2}{r!} H_r\big(x/\lambda\big)$$

5. Let $f \in \mathcal{S}$, $x \in \mathbb{R}$, and let $\big(P_t : t \in [0, +\infty]\big)$ be the Ornstein Uhlenbeck semigroup. Show that

$$\int_0^\infty e^{-2t} P_t f''(x) dt - x \int_0^\infty e^{-t} P_t f'(x) dt = f(x) - E\big(f(N)\big)$$

Hint: $\frac{d}{dt} P_t = L P_t$

6. On the Wiener space, compute the Malliavin derivative $DF \in H = L^2([0,T], dt)$

   (a) $F = W_T^n \quad n \in \mathbb{N}$,

   (b) $F = \exp(W_T)$

   (c) $F = \int_0^T W_s ds$

   (d) $F = f\big(\int_0^T g(s) dW_s\big)$, where $g \in L^2([0,T], dt)$ is deterministic with $\int_0^T g(s)^2 ds = 1$ and $f \in D^{1,2}(\mathbb{R}\gamma)$ with S obolev derivative $\partial f \in L^2(\gamma)$.

   (e) $F = \exp(W_T)$

7. On the Wiener space, let $g \in L^2([0,T], dt)$ deterministic write the chaos expansion of

$$\exp(\int_0^T g_s dW_s)$$