Open Quantum Systems: Exercise session 7

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Exercise 1: Relative entropy

The wave functions ψ describing a system and bath in contact with each other belong to the Hilbert space $\mathbb{H} = \mathbb{H}_S \otimes \mathbb{H}_B$. Consider the time evolution for a state ρ_0 of the system

$$\rho(t) = V(t)\rho_0 = \mathrm{Tr}_B \left(U(t,0)\rho_0 \otimes \rho_B U^\dagger(t,0)\right), \tag{1}$$

where the trace is taken over the bath and $U(t,0)$ is the unitary time evolution for the system and bath combined.

The relative entropy between two states ρ and ρ_0 of a system is defined in the following way

$$S(\rho|\rho_0) = \mathrm{Tr}(\rho \log \rho) - \mathrm{Tr}(\rho \log \rho_0). \tag{2}$$

This definition allows us to define the entropy production rate for the reduced system as

$$\sigma(\rho(t)) = -\frac{\mathrm{d}}{\mathrm{d}t} S(\rho(t)|\rho_0) \tag{3}$$

where $V(t)\rho_0 = \rho_0$ is stationary to the evolution. We want to prove that the entropy production is positive for the time evolution $V(t)$. To do this, first prove these general properties for the relative entropy:

(a) Show that for a unitary transformation U

$$S(U\rho U^\dagger | U\rho_0 U^\dagger) = S(\rho|\rho_0) \tag{4}$$

(b) In general one can prove that

$$S(\mathrm{Tr}_2(\sigma)| \mathrm{Tr}_2(\sigma_0)) \leq S(\sigma|\sigma_0) \tag{5}$$

here Tr_2 is a partial trace. Show that the equality holds when $\sigma = \rho \otimes \rho_B$ and $\sigma_0 = \rho_0 \otimes \rho_B$, i.e.

$$S(\rho \otimes \rho_B | \rho_0 \otimes \rho_B) = S(\rho|\rho_0) \tag{6}$$

(c) Use the properties above to show that the entropy production rate σ is positive for the time evolution $V(t)$.

Exercise 2: Resolvent

Consider the semi-classical Jaynes-Cummings Hamiltonian

$$H = \frac{\hbar\omega_{eg}}{2}\sigma_z + \frac{\hbar\Omega_0}{2}\left(\sigma_+ e^{-i\omega_{eg}t} + \sigma_- e^{i\omega_{eg}t}\right). \tag{7}$$

(a) Find a unitary transformation $U(t)$ that makes the Hamiltonian time independent, this is also called transforming to a rotating frame. Find the Hamiltonian $\bar{H}$ such that in the rotating frame

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi = -i\bar{H}\psi. \tag{8}$$

(b) Calculate the Fourier transform of

$$(i\frac{\mathrm{d}}{\mathrm{d}t} - \bar{H})R = \mathbb{I} \tag{9}$$

(c) Find the spectral decomposition of the resolvent $\hat{R}$, the Fourier transform of R, and calculate

$$\int_{C_+^\epsilon \cup C_-^\epsilon} \frac{\mathrm{d}z}{2\pi i} e^{-izt}\hat{R} \tag{10}$$

Exercise 3: Floquet theory

Consider a Hamiltonian of the form consisting of two parts

$$H(t) = H_0 + H_L(t), \tag{11}$$

where the time dependent part $H_L(t)$ is periodic with period $T = \frac{2\pi}{\omega_L}$. According to Floquet theory the solutions of the Schrödinger equation are of the form

$$\psi_r(t) = u_r(t)\exp(-i\epsilon_r/\hbar). \tag{12}$$

where the $u_r(t)$ are periodic with period T and are called the Floquet states.
(a) Show that

$$(H(t) - i\hbar\partial_t)u_r(t) = \epsilon_r u_r(t), \tag{13}$$

ϵ_r is also called the quasi-energy of the Floquet state.
(b) Express the time evolution operator $U_L(t, t_0)$ of the system in terms of the Floquet states.
(c) Show that the time evolution operator for the semi-classical Jaynes-Cummings Hamiltonian (7) can be written as

$$U_L(t, t_0) = U(t)\exp(i\bar{H}(t - t_0)\hbar)U^\dagger(t_0), \tag{14}$$

remember that $\bar{H}$ was defined by equation (8) and $U(t)$ is the transformation that makes the original Hamiltonian time independent. Use this to find the Floquet states for this system.

Exercise 4: Jaynes-Cummings revisited

Remember the full James-Cummings Hamiltonian from the lectures

$$\hat{H} = \frac{\hbar\omega_{eg}}{2}\sigma_z + \hbar\omega_c\left(a^\dagger a + \frac{1}{2}\right) + \frac{\hbar\Omega_0}{2}\left(\sigma_+ a + \sigma_- a^\dagger\right). \tag{15}$$

As was discussed in the lectures it possible to write the Hamiltonian as a diagonal matrix with $2x2$ block matrices labelled by n on the diagonal. The vectors acting on the n-th block matrix are of the form

$$\psi_n = a_n|g,n\rangle + b_n|e,n-1\rangle. \tag{16}$$

(a) What is $\psi_m(t)$ given that the initial state is $\psi_m(0) = |g,m\rangle$.

Imagine that the initial state at time t_0 is distributed with a probability P_m to be in the m-th subspace, i.e.

$$\phi_i = \sum_m \sqrt{P_m}\psi_m. \tag{17}$$

(b) Find the probability to be in the excited state at a later time t and the probability to have n photons at time t.
(c) Show that if the initial photons are Bose-Einstein distributed that

$$P_m = \frac{1}{1+\langle n\rangle}\left(\frac{\langle n\rangle}{1+\langle n\rangle}\right)^m \tag{18}$$

where $\langle n\rangle = [\exp(\hbar\omega/kT) - 1]^{-1}$ is the mean amount of photons.
Optional: (d) Make a plot of the probability for the qubit to be in the excited state in function of time for different values of $\langle n\rangle$ (e.g. 1, 50, 99).