Open Quantum Systems: Exercise session 5

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Exercise 1

Find the lowest eigen-frequency of cubic metallic cavity with the size $L = 1$ cm. (Model the cubic cavity as a three dimensional box)

Exercise 2

Prove the identity

$$e^{i\alpha\hat{\sigma}_x} = \begin{pmatrix} \cos\alpha & i\sin\alpha \\ i\sin\alpha & \cos\alpha \end{pmatrix}, \quad \text{where } \hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{1}$$

Exercise 3

A two level system driven by microwave signal is described by the Hamiltonian

$$\hat{H} = -\frac{\hbar\omega_{eg}}{2}\hat{\sigma}_z + \frac{\hbar\Omega_0}{2}\left(\hat{\sigma}_+ e^{-i\omega_{eg}t} + \hat{\sigma}_- e^{i\omega_{eg}t}\right) = \frac{\hbar}{2}\begin{pmatrix} -\omega_{eg} & \Omega_0 e^{-i\omega_{eg}t} \\ \Omega_0 e^{i\omega_{eg}t} & \omega_{eg} \end{pmatrix}. \tag{2}$$

We assume that at $t = 0$ the system is in the ground state, i.e. the system wave function has the form

$$\psi(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{3}$$

In order to find the wave function at finite time, one needs to solve the Schrödinger equation

$$i\hbar\frac{\partial\psi(t)}{\partial t} = \hat{H}\psi(t). \tag{4}$$

The solution has the form of a vector

$$\psi(t) = \begin{pmatrix} c_g(t) \\ c_e(t) \end{pmatrix}. \tag{5}$$

Find the occupation probabilities of the ground and excited states $P_g(t) = |c_g(t)|^2$ and $P_e(t) = |c_e(t)|^2$. At what times t one finds $P_g(t) = 0$, $P_e(t) = 1$? At what times equal occupations of the levels is achieved, i.e. $P_g(t) = P_e(t) = 1/2$?

Hint

Find the solution in the form $c_g(t) = e^{i\omega_{eg}t/2}u_g(t)$, $c_e(t) = e^{-i\omega_{eg}t/2}u_e(t)$.

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Exercise 4

A classical Josephson junction is described by the equation of motion

$$C\frac{\hbar\ddot{\phi}}{2e} + I_C \sin\phi = I, \tag{6}$$

the bias current is supposed to be smaller than the critical current I_C. At $I = 0$ the frequency of small oscillations around equilibrium $\phi = 0$ equals $\omega_p = \sqrt{2eI_C/\hbar C}$. Find this frequency at arbitrary bias current $I < I_C$.

Exercise 5

(problem 7.3 Nielsen and Chuang)
Let us look at the Jaynes-Cummings Hamiltonian for a single atom coupled to a single mode of an electromagnetic field

$$H = a^\dagger\sigma_- + a\sigma_+ \tag{7}$$

(1) For $u = e^{i\theta H}$, compute

$$A_n = \langle n|U|\alpha\rangle, \tag{8}$$

where $|n\rangle$ is a number eigenstate of the field and $|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$ is a coherent state, so $a|\alpha\rangle = \alpha|\alpha\rangle$ and $a^\dagger|\alpha\rangle = \left(\frac{\partial}{\partial\alpha} + \alpha^*\right)|\alpha\rangle$. A_n is an operator on atomic states and you should obtain

$$A_n = e^{-\frac{|\alpha|^2}{2}}\frac{\alpha^n}{n!}\begin{pmatrix} \cos(\theta\sqrt{n}) & \frac{i\sqrt{n}}{\alpha}\sin(\theta\sqrt{n}) \\ \frac{i\alpha}{\sqrt{n+1}}\sin(\theta\sqrt{n+1}) & \cos(\theta\sqrt{n+1}) \end{pmatrix} \tag{9}$$

(2) Consider the probability distribution

$$p_n = e^{-x}\frac{x^n}{n!} \tag{10}$$

Make a change of variables $n = x - L\sqrt{x}$ and use Sterling's approximation $n! \approx \sqrt{2\pi n}n^n e^{-n}$ to obtain

$$P_L \approx \frac{e^{-\frac{L^2}{2}}}{\sqrt{2\pi}} \tag{11}$$

(3) Now define $n = \alpha^2 + L\alpha$ and show for

$$c = \frac{1}{\alpha}\sqrt{\alpha^2 + L\alpha} \quad \text{and} \quad d = \frac{1}{\alpha}\sqrt{\alpha^2 + L\alpha + 1} \tag{12}$$

that

$$A_L \approx \frac{e^{-\frac{L^2}{4}}}{(2\pi)^{1/4}}\begin{pmatrix} \cos(\theta a/\alpha) & ia\sin(\theta a/\alpha) \\ \frac{i}{b}\sin(\theta b/\alpha) & \cos(\theta b/\alpha) \end{pmatrix} \tag{13}$$

Exercise 6

Exercise 5 from exercise session 4 (Partial Transpose Criterium)