INTRODUCTION TO OPEN QUANTUM SYSTEMS
EXERCISE SESSION 4

D. GOLUBEV, P. MURATORE-GINANNESCHI, K. SCHWIEGER

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1. One Fermion is a Qubit

Let $a \in \mathcal{B}(\mathcal{H})$ be an element satisfying the CAR, i. e., $a^2 = 0$ and $aa^* + a^*a = 1$. Show that $\mathcal{M} = \mathrm{lin}\{a, a^*, a^*a, aa^*\}$ is a subsystem with $\mathcal{M} \simeq M_2(\mathbb{C})$ by making explicit the map π that assigns to each matrix $X \in M_2(\mathbb{C})$ an operator $\pi(X) \in \mathcal{M}$. Check that π satisfies $\pi(XY) = \pi(X)\pi(Y)$ for all $X, Y \in M_2(\mathbb{C})$.

2. Computations with CAR

Let $a_1, \ldots, a_N \in \mathcal{B}(\mathcal{H})$ be operators satisfying the CAR, i. e., $a_i a_j = -a_i a_i$ and $a_i a_j^* + a_j^* a_i = \delta_{i,j}$ for all $1 \le i, j \le N$. We write $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ for the *-subalgebra generated by $a_1, \ldots, a_N$. Show the following statements:

(1) The operators $b_i := a_i^*$ also satisfy the CAR.
(2) Each a_i and all products $V := a_{i_1} \ldots a_{i_k}$ are partial isometries with $V^2 = 0$.
(3) All initial and final projections $a_i^* a_i$ and $a_j a_j^*$ commute among each other.
(4) The orthogonal projection onto the intersection of the subspace $a_1 a_1^* \mathcal{H}$ and $a_2 a_2^* \mathcal{H}$ is an operator in $\mathcal{M}$. Write the projection in terms of a_1 and a_2.
(5) $\mathcal{M}$ is the linear span of all products of the form
$$a_{i_1} \ldots a_{i_n} a_{j_m}^* \ldots a_{j_1}^*, \tag{1}$$
where we agree that the product of length 0 is 1. For which combination of indices does such a product vanish?

3. Fermions and Independence

Consider $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$ with the two Fermionic annihilation operators $a_1 = a \otimes 1$ and $a_2 = \sigma_z \otimes a$, where $a = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ and $\sigma_z = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Let $\mathcal{M}_k = \mathrm{lin}\{a_k, a_k^*, a_k^* a_k, a_k a_k^*\}$ for $k \in \{1, 2\}$ be the associated subsystems.

(1) Check that $\mathcal{M}_1$ and $\mathcal{M}_2$ are not each others commutant and determine the commutant of $\mathcal{M}_2$.
(2) Show that $\mathcal{M}_1$ and $\mathcal{M}_2$ are independent with respect to the vacuum state.

In applications each Fermionic mode has an associated energy ϵ_1, ϵ_2. Consider the *free Hamiltonian* $H = \epsilon_1 a_1^* a_1 + \epsilon_2 a_2^* a_2$ and the density matrix $\rho_\beta := e^{-\beta H} / \operatorname{Tr}(e^{-\beta H})$, called the *Gibbs state* of *inverse temperature* $\beta > 0$.[1]

(3) Show that $\mathcal{M}_1$ and $\mathcal{M}_2$ are independent with respect to any Gibbs state.

The following exercise is optional but—I think—interesting:

(4) Determine all density matrices $\rho \in \mathcal{B}(\mathcal{H})$ such that $\mathcal{M}_1$ and $\mathcal{M}_2$ are independent with respect to ρ.

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[1] You may want to put some $\hbar$'s here if you are a physicist.

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4. Computing the Schmidt Decomposition

(1) Determine the Schmidt decomposition of the following (not normalized) vector in $\mathbb{C}^4 = \mathbb{C}^2 \otimes \mathbb{C}^2$:

$$\psi := (1+\sqrt{2},\ 1-\sqrt{2},\ 1-\sqrt{2},\ 1+\sqrt{2})^T.$$

(2) Consider the tensor product $\mathcal{H}_S \otimes \mathcal{H}_E$ and let $\varphi, \psi \in \mathcal{H}_S \otimes \mathcal{H}_E$ be two unit vectors with $\mathrm{Tr}_2(|\varphi\rangle\langle\varphi|) = \mathrm{Tr}_2(|\psi\langle\rangle\psi|)$. Show that there is a partial isometry $V \in \mathcal{B}(\mathcal{H}_E)$ with $\psi = (1 \otimes V)\varphi$. Moreover, V can be choosen unitary if $\mathcal{H}_E$ is finite-dimensional.

5. Partial Transpose Criterium

Consider the tensor product $M_N(\mathbb{C}) \otimes M_N(\mathbb{C})$ of two matrix algebras. For a block matrix $X = \begin{pmatrix} X_{11} & \dots & X_{1N} \\ \vdots & & \vdots \\ X_{N1} & \dots & X_{NN} \end{pmatrix}$ with $X_{ij} \in M_N(\mathbb{C})$ its partial transpose is definde as

$$\mathrm{PT}(X) := \begin{pmatrix} X_{11} & \dots & X_{N1} \\ \vdots & & \vdots \\ X_{1N} & \dots & X_{NN} \end{pmatrix}.$$

(1) Show that the partial transpose $\mathrm{PT}(\rho)$ of a separable density matrix ρ is again a density matrix, in particular it is positive.

The set $M_N(\mathbb{C}) \otimes M_N(\mathbb{C})$ has a particularly simple symmetry given by the tensor flip F, i. e., the unitary $F \in M_N(\mathbb{C}) \otimes M_N(\mathbb{C})$ given by $F(\varphi_1 \otimes \varphi_2) = \varphi_2 \otimes \varphi_1$ for all $\varphi_1, \varphi_2 \in \mathbb{C}^N$.

(1) Write F as a block matrix.
(2) Show that $P_+ := (1+F)/2$ is the projection onto the space of fixed points of F, called the *symmetric subspace*. Show that $P_- := (1-F)/2$ is the projection onto the space of vectors ψ with $F\psi = -\psi$, called the *antisymmetric subspace*.
(3) What is the dimension d_+ of the symmetric subspace and d_- of the antisymmetric subspace?

States of the form $\rho(\alpha) = \alpha(P_+/d_+) + (1-\alpha)(P_-/d_-)$ with $0 \le \alpha \le 1$ are called *Werner states*.

(1) Apply the criterium in point 1 to find a range of parameters α for which $\rho(\alpha)$ is entangled?

6. Entanglement and a Question of Locality?

Let e, f by a fixed orthonormal basis of $\mathbb{C}^2$ and consider the vector

$$\psi = \tfrac{1}{\sqrt{2}}(e \otimes e + f \otimes f)$$

describing the state of the quantum system $M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$. Now let $P \in M_2(\mathbb{C})$ be an *arbitrary* one-dimensional projection and consider a measurement of P in the first tensor factor, i. e., a measurement of $P \otimes 1$, which yields either 1 or 0 as a result.

(1) What is the probability to obtain the measure result 1 / 0? In each case, what is the state after the measurement?
(2) Compare the marginal density on the second tensor factor before and after the measurement. Do you find the result surprising?