INTRODUCTION TO OPEN QUANTUM SYSTEMS
EXERCISE SESSION 3

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1. Position and Momentum Representation

Consider the Hilbert space $\mathcal{H} = L^2(\mathbb{T})$ and denote by $F : \ell^2(\mathbb{Z}) \to L^2(\mathbb{T})$ the Fourier transform. We consider the position operator

$$Q : L^2(\mathbb{T}) \to L^2(\mathbb{T}), \quad (Q\varphi)(z) := z\,\varphi(z).$$

(1) Check that Q is a unitary operator. How does the position operator look like in the Fourier-transformed picture? That is, compute $F^{-1}QF$ explicitly.

The representation of operator on $L^2(\mathbb{T})$ is called *position representation*, the Fourier transformed representation on $\ell^2(\mathbb{Z})$ is called *momentum representation*.

(2) What is the spectrum of Q? (For the answer there is no detailed argument requested, but optionally: Prove that your answer is correct.) For a given subinterval E of the spectrum, what is the corresponding spectral subspace?

2. Momentum Operator on a Circle

As in the previous exercise consider the Hilbert space $\mathcal{H} = L^2(\mathbb{T})$ and the Fourier transform $F : \ell^2(\mathbb{Z}) \to L^2(\mathbb{T})$.

(1) The momentum operator is conveniently defined on $\ell^2(\mathbb{Z})$ by $\tilde{P}(\psi_n)_n := (n\,\psi_n)_n$ for suitable vectors $(\psi_n)_n \in \ell^2(\mathbb{Z})$.[1] What is a dense linear subset of vectors on which P can be defined.
(2) Show that $\tilde{P}$ is not bounded on your selected subspace.

For the rest of this exercise we ignore possible problems stemming to the fact that $\tilde{P}$ is unbounded.

(3) What is the spectrum of $\tilde{P}$ and what are corresponding eigenvectors?
(4) How does the momentum operator look like in position representation? That is, compute $P := F\tilde{P}F^{-1}$. What commutation relations do P and Q satisfy?

3. Orthogonal Projections

Show that for an operator $P \in \mathcal{B}(\mathcal{H})$ the following statements are equivalent:

(a) P is an orthogonal projection, i. e., $P^2 = P = P^*$.
(b) $P^2 = P$ and $\ker P \perp P\mathcal{H}$.
(c) $P = P^*$ and P has spectrum in $\{0, 1\}$.

Optional: Show that the conditions are also equivalent to

(d) $P^2 = P$ and $\|P\|_{\mathrm{op}} \leq 1$.

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[1] Here we consider units in which the Planck constant is normalized to $\hbar = 1$.

Thursday, 29. Sep.

4. Partial Isometries

(1) Let $V \in \mathcal{B}(\mathcal{H})$ be an operator such that V^*V is a projection. Show that also VV^* is a projection.
Hint: A way to proceed is to first show that $P \in \mathcal{B}(\mathcal{H})$ is a projection if $P^* = P$ and $P^3 = P^2$.

(2) Let $V \in \mathcal{B}(\mathcal{H})$ be a partial isometry. Describe in word in terms of V onto which subspace the projections V^*V and VV^* project.
V^*V is called the *initial projection* and VV^* is called the *final projection.*

5. Weak Convergence

For the weak convergence you may wonder what operations are weakly continuous and which are not.

(1) Let $A_i, A, B \in \mathcal{B}(\mathcal{H})$. Check that, if $A_i \to A$ weakly then $A_i^* \to A$, $BA_i \to BA$, and $A_iB \to AB$ weakly.

(2) Let $\mathcal{H} = \ell^2(\mathbb{N})$ and $S : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ the one-sided shift. Show that the sequence $S^n \to 0$ weakly but not in norm.

(3) It follows from 1, 2 that also $A_n := S^n$ and $B_n := (S^*)$ converge to zero weakly. Show that A_nB_n converges weakly to zero but B_nA_n does not.

6. Tensor Products

Let $\mathcal{H}$ be an arbitrary Hilbert space. We write $L^2(\mathbb{T}, \mathcal{H})$ for the set of all functions (up to measure zero) $\varphi : \mathbb{T} \to \mathcal{H}$ with a finite integral $\int_{\mathbb{T}} |\varphi(z)|^2 \, dz$. Define an inner product on $L^2(\mathbb{T}, \mathcal{H})$ and show that

$$L^2(\mathbb{T}, \mathcal{H}) = L^2(\mathbb{T}) \otimes \mathcal{H}$$