Open Quantum Systems: Exercise session 2

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September 15, 2016

Tuesday 20/09

Exercise 1

A function $f : \mathbb{R} \to \mathbb{C}$ is called positive if

$$\sum_{i,j}^{n} \xi_i^* f(t_i - t_j)\xi_j \geq 0$$

for all $\xi_1, \dots, \xi_n \in \mathbb{C}$ and all $t_1, \dots, t_n \in \mathbb{R}$.
(a) Show that $f(0) \geq 0$ for all $t \in \mathbb{R}$
(b) Show that $f(t) = f^*(-t)$ and $f(0) \geq |f(t)|$
(c) Show that f is positive if and only if for all $t_1, \dots, t_n \in \mathbb{R}$ the matrix

$$X := (f(t_i - t_j))_{1 \leq i,j \leq n}$$

is positive semi-definite.
Hint: for (a) take $n = 1$ and for (b) take $n = 2$

Exercise 2

In physics a system described by the Hilbert space [1] $\mathbb{C}^2$ is called a qubit. Show that the density matrix of a qubit ρ can always be written in the form

$$\rho = \frac{1}{2}(\mathbb{I} + v \cdot \sigma)$$

where $v \in \mathbb{R}^3$, $\|v\| \leq 1$ and $v \cdot \sigma = v_1\sigma_x + v_2\sigma_y + v_3\sigma_z$. Show that $\|v\| = 1$ if and only if the qubit is in a pure state.

Exercise 3

From the last exercise we can see that the pure states of the qubit represent points on the unit sphere in $\mathbb{R}^3$, which in this context is called the Bloch sphere.
(a) Show that every unit vector of the qubit can be written as

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\phi}|1\rangle,$$

[1] with the inner product $\langle\phi|\psi\rangle = \phi_1^*\psi_1 + \phi_2^*\psi_2$

where $|0\rangle$, $|1\rangle$ is a fixed orthonormal basis.
The two angles (ϕ, θ) are the coordinates of the pure states on the Bloch sphere in the given basis.
(b) Show that orthonormal bases correspond to antipodal points on the Bloch sphere.
(c) Express the transition probability between to states of the qubit as a geometric property of the corresponding points on the Bloch sphere.

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Exercise 4

(Ballentine, exercise 2.5)
Which of the following are state operators? Are they pure states? If so decompose them in pure unit vectors.

$$\rho_1 = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{3}{4} \end{pmatrix}, \quad \rho_2 \begin{pmatrix} \frac{9}{25} & \frac{12}{25} \\ \frac{12}{25} & \frac{16}{25} \end{pmatrix}$$
$$\rho_3 = \frac{1}{3}|u\rangle\langle u| + \frac{2}{3}|v\rangle\langle v| + \frac{\sqrt{2}}{3}|v\rangle\langle u| + \frac{\sqrt{2}}{3}|u\rangle\langle v|,$$

where $\langle u|u\rangle = \langle v|v\rangle = 1$ and $\langle u|v\rangle = 0$,

$$\rho_4 = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{4} \\ 0 & \frac{1}{2} & 0 \\ \frac{1}{4} & 0 & 0 \end{pmatrix}, \quad \rho_5 = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{4} \\ 0 & \frac{1}{4} & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} \end{pmatrix}.$$

Exercise 5

The exponential function for a square matrix $A \in M_n(\mathbb{C})$ is defined as

$$\exp(A) = \sum_{n=0}^{\infty} \frac{1}{n!} A^n.$$

(a) Show that for $v \in \mathbb{C}^3$ a unit vector, θ a real number and $v \cdot \sigma = v_1\sigma_x + v_2\sigma_y + v_3\sigma_z$:

$$\exp(i\theta v \cdot \sigma) = \cos(\theta)\mathbb{I} + i\sin(\theta) v \cdot \sigma$$

(b) Define the operator $\mathcal{L}_A$ that acts on $X \in M_n(\mathbb{C})$ as

$$\mathcal{L}_A X = [A, X] = AX - XA.$$

The exponential of this operator is defined as

$$\exp(\mathcal{L}_A)X := \sum_{n=0}^{\infty} \frac{\mathcal{L}_A^n}{n!} X,$$

where $\mathcal{L}_A^n X = \underbrace{\mathcal{L}_A \circ \mathcal{L}_A \circ \ldots \mathcal{L}_A}_{\text{n-times}} X$. Show then that

$$\exp(A)X\exp(-A) = \exp(\mathcal{L}_A)X.$$

Exercise 6

(Ballentine, exercise 2.9)
Let $R = \begin{pmatrix} 6 & -2 \\ -2 & 9 \end{pmatrix}$ represent a dynamical variable and $\psi = \begin{pmatrix} a \\ b \end{pmatrix}$ be an arbitrary unit vector. Calculate $\langle R^2 \rangle := \langle \psi | R^2 | \psi \rangle$ in two ways:
(a) Evaluate $\langle \psi | R^2 | \psi \rangle$ directly.
(b) Find the eigenvalues and eigenvectors of R

$$R|r_n\rangle = r_n|r_n\rangle$$

and expand the state vector as a linear combination of the eigenvectors,

$$|\psi\rangle = c_1|r_1\rangle + c_2|r_2\rangle$$

then evaluate $\langle R^2 \rangle = r_1^2|c_1|^2 + r_2^2|c_2|^2$