Open Quantum Systems: Exercise session 1

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Tuesday 13/10

Exercise 1

(a) Prove that the trace of an operator A, $\operatorname{Tr} A = \sum_n \langle u_n|A|u_n\rangle$ is independent of the orthonormal basis $\{|u_n\rangle\}$ chosen for its evaluation.
(b) Show that $\operatorname{Tr}(A|\psi\rangle\langle\psi|) = \langle\psi|A|\psi\rangle$.

Exercise 2

(Ballentine, exercise 1.3)
Consider the vector space $M_n(\mathbb{C})$, show that the Pauli matrices form a basis for this vector space:

$$\mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Exercise 3

(Ballentine, exercise 1.4)
(a) If A and B are matrices of the same shape, show that $(A, B) = \operatorname{Tr}(A^\dagger B)$ has all the properties of an inner product. How about $(A, B)' = \operatorname{Tr}(\rho A^\dagger B)$, for ρ a density matrix?
(b) Show that the Pauli matrices (see exercise 3) are orthogonal with respect to the inner product (A, B).

Thursday 15/10

Exercise 4

(Nielsen and Chuang, exercise 2.57)
Suppose $\{L_l\}$ and $\{M_m\}$ are two sets of measurement operators. Show that a measurement defined by the measurement operators $\{M_m\}$ followed by a measurement defined by the measurement operators $\{L_l\}$ is physically equivalent to a single measurement defined by the measurement operators $\{N_{ml}\}$ with

$N_{ml} \equiv L_l M_m$. More precisely show that the probability of consecutive measurements is equal to the probability of the measurement N_{ml} and that the resulting states are equal.

Exercise 5

Prove that for $X \in M_n(\mathbb{C})$ the following statements are equivalent

a. X is of the form $B^\dagger B$, with $B \in M_n(\mathbb{C})$.

b. $\langle \phi | X | \phi \rangle \geq 0 \ \forall \phi \in \mathbb{C}^n$.

c. $A = A^\dagger$ and every eigenvalue is non-negative.

Exercise 6

(a) Show that for A, $B \in M_n(\mathbb{C})$ the trace is cyclic i.e. $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$.
(b) Consider X, $Y \in M_n(\mathbb{C})$ and X, $Y \geq 0$, show that $\mathrm{Tr}(XY) \geq 0$.
(c) Is XY a positive matrix?
hint: for (b) use (a) and Exercise 5

Exercise 7

For a particle in one dimension the momentum operator P and the position operator Q satisfy the commutation relation

$$[P, Q] = \frac{\hbar}{i}\mathbb{I}.$$

By taking the trace on the left hand side one finds that $\mathrm{Tr}[P, Q] = 0$ while on the left hand side we can see that $\mathrm{Tr}(\frac{\hbar}{i}\mathbb{I}) \neq 0$. What is going on?