Open Quantum Systems: Exercise session 6

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Exercise 1: Invariance of the Lindblad form

Show that the Lindblad equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = -i[H, \rho(t)] + \sum_k \gamma_k \left(A_k \rho_S(t) A_k^\dagger - \frac{1}{2}\{A_k^\dagger A_k, \rho_S(t)\} \right) \tag{1}$$

is invariant under:

1. Unitary transformations on the effect operators

$$\sqrt{\gamma_k'} A_k' = \sum_l u_{kl} \sqrt{\gamma_l} A_l \tag{2}$$

   where the matrix U with entries u_{ij} is a unitary matrix.

2. Transformations of the form

$$A_k' = A_k + a_k \tag{3}$$

$$H' = H + \frac{1}{2i} \sum_j \gamma_j \left(a_j^* A_j - a_j A_j^\dagger \right) + b \tag{4}$$

   where the a_i are complex numbers and b is real.

Exercise 2: Multi-time correlation functions

Suppose we have a density ρ_0 evolving as $\rho(t) = V(t,0)\rho_0$, with $\frac{\mathrm{d}}{\mathrm{d}t}V(t,0) = \mathcal{L}V(t,0)$, and a set of operators B_i, such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle B_i(t)\rangle = \sum_j G_{ij} \langle B_j(t)\rangle \tag{5}$$

with some coefficient matrix G_{ij}.
Show that then

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\langle B_i(t+\tau) B_l(t)\rangle = \sum_j G_{ij} \langle B_i(t+\tau) B_l(t)\rangle. \tag{6}$$

Hint: remember that $\langle B_i(t+\tau)B_l(t)\rangle = tr(B_i V(t+\tau, t) B_l V(t,0)\rho_0)$.

Exercise 3: Qubit

The Lindblad equation, in the interaction picture, of a driven qubit in a radition field is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = \frac{i\Omega}{2}[\sigma_+ + \sigma_-, \rho(t)] + \gamma_0(N+1)\left(\sigma_-\rho(t)\sigma_+ - \frac{1}{2}\{\sigma_+\sigma_-\rho(t)\}\right)$$
$$+ \gamma_0 N\left(\sigma_+\rho(t)\sigma_- - \frac{1}{2}\{\sigma_-\sigma_+\rho(t)\}\right) \tag{7}$$

the first term on the rhs is due to the drive.

1. Show that the Bloch vector $\langle\vec{\sigma}(t)\rangle = (\langle\sigma_x(t)\rangle, \langle\sigma_y(t)\rangle, \langle\sigma_z(t)\rangle)$ satisfies a differential equation of the from

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\vec{\sigma}(t)\rangle = G\langle\vec{\sigma}(t)\rangle + b \tag{8}$$

   where G is a matrix and b is a vector. Calculate the stationary solution $\langle\vec{\sigma}\rangle_s$.

2. Show that the vector $\langle\langle\vec{\sigma}(t)\rangle\rangle = \langle\vec{\sigma}(t)\rangle - \langle\vec{\sigma}\rangle_s$ satisfies the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\langle\vec{\sigma}(t)\rangle\rangle = G\langle\langle\vec{\sigma}(t)\rangle\rangle. \tag{9}$$

   Use this to find the solution for $\langle\sigma_z(t)\rangle$ in terms of the initial condition $\langle\vec{\sigma}(0)\rangle = \langle\vec{\sigma}_0\rangle$.

3. Find the solution of $\langle\sigma_z(t)\sigma_-(0)\rangle_s$, with the average taken over the stationary distribution.

Hint: for 3. try to use the result of Exercise 2