Open Quantum Systems: Exercise session 8

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November 9, 2016

Intro: a qubit interacting with a bath

We suppose that the bath is described by a free electromagnetic field. The Hamiltonian for the bath is therefore a series ranging over the field modes

$$\hat{H}_E = \sum_k \omega_k \hat{b}_k^\dagger \hat{b}_k \tag{1}$$

The integer k codes all of the information specifying each mode: its frequency, direction, transverse structure and polarization. The annihilation and creation operators for each mode are independent and they obey the Bosonic commutation relations

$$[\hat{b}_k \, , \hat{b}_l^\dagger] = \delta_{kl} \tag{2}$$

The Hamiltonian of the qubit (two-level "atom") is

$$\hat{H}_a = \frac{\omega_a}{2} \hat{\sigma}_z \tag{3}$$

Here ω_a is the energy difference between the ground $|g\rangle$ and excited $|e\rangle$ states, and

$$\hat{\sigma}_z \equiv \hat{\sigma}_3 = |e\rangle\langle e| - |g\rangle\langle g| = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{4}$$

is the "inversion operator" for the qubit. Recall that $\hat{\sigma}_z$ together with

$$\hat{\sigma}_x \equiv \hat{\sigma}_1 = |e\rangle\langle g| + |g\rangle\langle e| = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \& \qquad \hat{\sigma}_y \equiv \hat{\sigma}_2 = -\imath |e\rangle\langle g| + \imath |g\rangle\langle e| = \begin{bmatrix} 0 & -\imath \\ \imath & 0 \end{bmatrix} \tag{5}$$

are Pauli matrices satisfying

$$\hat{\sigma}_j \hat{\sigma}_k = \delta_{jk} \mathsf{I} + \imath \, \epsilon_{jkl} \, \hat{\sigma}_l \tag{6}$$

for I the identity matrix.

The coupling of the electromagnetic field to a qubit can be described by the so-called dipole-coupling Hamiltonian

$$\hat{V} = \sum_k g_k (\hat{b}_k + \hat{b}_k^\dagger) \hat{\sigma}_x \tag{7}$$

The *real* coefficient g_k is proportional to the dipole matrix element for the transition.

The total Hamiltonian operator is therefore

$$\hat{H} = \hat{H}_E + \hat{H}_a + \hat{V} \tag{8}$$

Question 1

1. Show that the density (or state matrix) of the qubit can always be written as

$$\hat{\rho}(t) = \frac{1}{2}\left(\mathsf{I} + x(t)\hat{\sigma}_x + y(t)\hat{\sigma}_y + z(t)\hat{\sigma}_z\right) \tag{9}$$

where $x(t), y(t), z(t)$ are real numbers. What is the physical interpretation of these numbers ?

2. Show that

$$x^2(t) + y^2(t) + z^2(t) \leq 1 \tag{10}$$

with equality holding if and only if $\hat{\rho}$ describes a pure state.

Question 2

1. Prove that in the interaction picture

$$\hat{H} = \sum_k g_k \left(\hat{b}_k e^{-\imath\,\omega_k\,t} + \hat{b}_k^\dagger e^{\imath\,\omega_k\,t}\right)\left(\frac{\sigma_x - \imath\,\sigma_y}{2} e^{-\imath\,\omega_a t} + \frac{\sigma_x + \imath\,\sigma_y}{2} e^{\imath\,\omega_a t}\right). \tag{11}$$

2. What is the equation for the density of states (state matrix) in the interaction picture?

Question 3

Adopt the interaction picture.

Assume that the *Born (or weak coupling) approximation holds*: the qubit affects the bath only in a negligible way.

- How do you mathematically formulate such an assumption?

Assume that the *rotating wave approximation holds.*

- Argue that this latter assumption mathematically transduces into hypothesizing that

$$\hat{H} \approx \sum_k g_k \left(\hat{b}_k \hat{\sigma}_+ \, e^{-\imath\,(\omega_k - \omega_a)\,t} + \hat{b}_k^\dagger \, \hat{\sigma}_- \, e^{\imath\,(\omega_k - \omega_a)\,t}\right) \tag{12}$$

Show that under these approximations the equation governing the evolution of the density matrix of the system is

$$\dot{\rho}(t) = -\int_0^t \mathrm{d}t_1 \left\{\Gamma(t - t_1)\left[\hat{\sigma}_+ \, \hat{\sigma}_- \, \rho(t_1) - \hat{\sigma}_- \, \rho(t_1) \, \hat{\sigma}_+\right] + \text{Hermitian conjugate}\right\} \tag{13}$$

and

$$\Gamma(t) = \sum_k g_k^2 \, e^{-\imath(\omega_k - \omega_a)\,t} \tag{14}$$

Assume that the *Markov approximation holds*: there is an infinite number of modes, whilst the coupling constant scales as

$$g_k = O(V_k^{-1/2}) \tag{15}$$

if V_k is the physical volume of mode k. Argue that if V_k tends to infinity it is legitimate to estimate (14) as

$$\Gamma(t) = \int_0^{\infty} \mathrm{d}\omega \, g^2(\omega) \, n(\omega) \, e^{-\imath(\omega-\omega_a)\,t} \tag{16}$$

where $n(\omega)$ is the density of field modes as a function of frequency and

$$f(\omega) = g^2(\omega) \, n(\omega) \tag{17}$$

is a function of frequency smoothly varying for $\omega \approx \omega_a$.

- Assume that

$$f(\omega) = \begin{cases} \text{constant} & \text{if } \omega \in [0\,, 2\,\omega_a] \\ 0 & \text{if } \omega > 2\,\omega_a \end{cases} \tag{18}$$

  show that $\Gamma(t)$ is sharply peaked around $t = 0$.

- Argue that whenever $f(\omega)$ is a smoothly varying function of frequency for $\omega \approx \omega_a$ $\Gamma(t)$ is also sharply peaked around $t = 0$.

- Use these facts to show that the master equation for the qubit density of states reduces to

$$\dot{\rho} = -\imath \, \frac{\Delta\,\omega_a}{2} \, [\hat{\sigma}_z \,, \rho] + \gamma \, \hat{\sigma}_- \, \rho \, \sigma_-^{\dagger} - \frac{\gamma}{2} \left(\hat{\sigma}_-^{\dagger} \, \hat{\sigma}_- \, \rho + \rho \, \hat{\sigma}_-^{\dagger} \, \hat{\sigma}_- \right) \tag{19}$$

  where the real parameters $\Delta\,\omega_a$ (radiative shift) and γ (radiative decay rate) satisfy the relation

$$\Delta\,\omega_a - \imath \frac{\gamma}{2} = -\imath \int_0^{\infty} \mathrm{d}t \, \Gamma(t) \tag{20}$$