Open Quantum Systems: Exercise session 5

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Exercise 1: Kraus Decomposition

Consider the map $T := \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H}) : \rho \mapsto \mathrm{Tr}(\rho)\rho_0$ for $\rho_0 \in \mathcal{L}(\mathcal{H})$.
(a) Show that it is completely positive.
(b) Find the Kraus representation, find $V_i \in \mathbb{B}(\mathcal{H}))$ so that the map can be written as $T(\rho) = \sum_i V_i \rho V_i^\dagger$ with $\sum_i V_i^\dagger V_i = \mathbb{I}$ the identity operator.

Exercise 2: Choi encoding

For which values of λ the following maps are completely positive?

$$T\left(\begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}\right) = \begin{pmatrix} \rho_{11} & \lambda\rho_{12} \\ \lambda\rho_{21} & \rho_{22} \end{pmatrix} \tag{1}$$

$$T(\rho) = \lambda\rho + (1-\lambda)\frac{\mathbb{I}}{2}\,\mathrm{Tr}(\rho) \tag{2}$$

$$T(\rho) = \lambda\rho + (1-\lambda)\rho^T \tag{3}$$

Exercise 3: Transpose map

The transpose $T(A) = A^T$ for $A \in M_2(\mathbb{C})$ is not completely positive
(a) Find a counterexample to show that the transpose map is indeed not completely positive.
(b) Show that the Kadison inequality does not hold for the transpose map.

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Exercise 4

Consider two qubits interacting by the Hamiltonian

$$H = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{4}$$

where the ground state is represented by $|g\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

(a) Compute the completely positive map

$$T(\rho) = \mathrm{Tr}_2 \left(e^{-iHt} \rho \otimes \sigma e^{iHt} \right), \tag{5}$$

where $\sigma = \frac{1}{2}|e\rangle\langle e| + \frac{1}{2}|g\rangle\langle g|$.
(b) Now suppose that we let the two qubits evolve from the initial state $\rho \otimes \sigma$ to a time t and then we perform an inderect measurement on the first qubit by measuring whether the second qubit is in the ground state. What are the possible outcomes of this measurement and their probabilities?
(c) Suppose now that after the indirect measurement we forget the results, compare this with the answer from (a).

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Exercise 6

Consider a map

$$\Lambda \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} = \begin{pmatrix} -a\rho_{11} + b\rho_{22} & c\rho_{12} \\ \bar{c}\rho_{21} & a\rho_{11} - b\rho_{22}, \end{pmatrix} \tag{6}$$

with $a, b \geq 0$ and $a + b \leq -c - \bar{c}$.
(a) Compute

$$e^{t\Lambda} \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}. \tag{7}$$

(b) What is the long time evolution of this system, i.e. consider the limit $t \leftarrow \infty$.