Open Quantum Systems: Exercise session 11

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Exercise 1

We have a stochastic differential equation of the form

$$\mathrm{d}\xi_t = -\frac{\xi_t}{\tau}\,\mathrm{d}t + \beta\xi_t\,\mathrm{d}\nu_t \tag{1}$$

where ν_t is a Poisson process, the average of its short time increment is $\mathbb{E}(\mathrm{d}\nu_t) = \gamma\,\mathrm{d}t$.

1. Take $\xi_t = e^{-\frac{t}{\tau}}\chi_t$ and use this to find a solution for the stochastic differential equation (1).
2. Let $\mathcal{L}_{\xi_t}$ be the backward generator of the stochastic process. Taking the derivative of the average of a function f is then given by:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}(f(\xi_t)) = \mathbb{E}(\mathcal{L}_{\xi_t}(f(\xi_t))). \tag{2}$$

   Use this to show that the probability density of the stochastic process $p(x,t)$ satisfies the differential equation

$$\partial_t p(x,t) = \mathcal{L}^\dagger p(x,t), \tag{3}$$

   where $\mathcal{L}^\dagger$ is the adjoint of $\mathcal{L}$.

Exercise 2: Heterodyne detection

In this exercise we derive the stochastic Schrödinger equation for a driven qubit in contact with an electromagnetic field. The electromagnetic field is in the vacuum state, due to the interaction a photon can be created which is absorbed by a detector. The derivation is similar to the case of homodyne detection discussed in the lecture on 8/12.

The full Hamiltonian is given by

$$H = H_S + H_d(t) + H_E + H_I \tag{4}$$

$$= \hbar\omega a^\dagger a - \frac{\Omega}{2}(a^\dagger e^{-i\omega t} + a e^{i\omega t}) + \sum_k \omega_k b_k^\dagger b_k + \sum_k (c_k a^\dagger b_k + c_k^* a b_k^\dagger) \tag{5}$$

where a and $a^\dagger$ are the fermionic ladder operators, b_k and $b_k^\dagger$ are bosonic ladder operators and the $c_k \in \mathbb{C}$.

1. Calculate the Hamiltonian in the interaction picture:

$$\bar{H} = e^{i(H_E+H_S)t}\,(H_d(t) + H_I)\,e^{-i(H_E+H_S)t} \tag{6}$$

2. Let's define the collapse operators

$$r_k = \langle k|U_I(t)(|0\rangle \otimes |\psi\rangle) \tag{7}$$

   where $|k\rangle = b_k^\dagger|0\rangle$ is the state with one photon of the k-th mode and $U_I(t)$ is the time evolution operator in the interaction picture. These operators give the possible jumps that the qubit can make that result in a photon in the k-th mode being created. Calculate these operators by expanding $U_I(t)$ up to second order in Ω and c_k.

3. The jump rate is defined by

$$\Gamma(t) = \frac{1}{\|a|\psi\rangle\|^2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_k \|r_k|\psi\rangle\|^2. \tag{8}$$

   Calculate $\gamma = \lim_{t\uparrow\infty}\Gamma(t)$ by approximating the field as a continuum of field modes and taking $t \gg \omega^{-1}$.

4. Calculate

$$r_0 = \langle 0|U_I(t)(|0\rangle \otimes |\psi\rangle) \tag{9}$$

   under the same assumptions as in 2. and keeping only first order terms in *Omega*. This is the action on the qubit when no photon has been measured. Show that

$$\|\langle 0|U_I(t)(|0\rangle \otimes |\psi\rangle)\|^2 = 1 - \gamma\|a\psi\|^2. \tag{10}$$

5. Use all of the above to write down the stochastic Schrödinger equation for the qubit.