HARMONIC ANALYSIS
2016

8. Homework sheet
17.11.2016

8.1. **Homework.** Let $f \in BMO(\mathbb{R}^n)$. Prove that

$$\frac{f(x)}{(1+|x|)^{n+1}} \in L^1(\mathbb{R}^n)\,.$$

Hint: Show that

$$\int_{\mathbb{R}^n} \frac{|f(x) - f_{B_1(0)}|}{(1+|x|)^{n+1}}\, dx \le c_n ||f||_{BMO(\mathbb{R}^n)}\,.$$

8.2. **Homework.** For a cube Q in $\mathbb{R}^n$ let us define

$$BMO(Q) := \Big\{ f \in L^1_{\text{loc}}(Q) \,:\, \sup_{R \subset Q} \frac{1}{|R|} \int |f(x) - f_R|\, dx < \infty \Big\}$$

where the supremum is taken over all cubes R in Q.
Let Q be fixed and $f \in L^1_{\text{loc}}(Q)$. If for any cube R in Q there exists $a_R \in \mathbb{R}$ such that

$$\big|\big\{ x \in R \,:\, |f(x) - a_R| > \lambda \big\}\big| \le b|R| \exp(-a\lambda)$$

with some absolute constants $a > 0$ and $b > 0$ for all $\lambda > 0$ then, show that $f \in BMO(Q)$.

8.3. **Homework.** Let $1 < p < \infty$. Prove that

$$\sup_Q \left(\frac{1}{|Q|} \int_Q |f(x) - f_Q|^p \, dx \right)^{1/p} \approx ||f||_{BMO(\mathbb{R}^n)}$$

where the supremum is taken over all cubes in $\mathbb{R}^n$.

8.4. **Homework.** Let $0 < p < q < \infty$ and $f \in L^p(\mathbb{R}^n) \cap BMO(\mathbb{R}^n)$. Show that $f \in L^q(\mathbb{R}^n)$ and

$$||f||_{L^q(\mathbb{R}^n)} \le c_{n,p,q} ||f||^{p/q}_{L^p(\mathbb{R}^n)} ||f||^{1-p/q}_{BMO(\mathbb{R}^n)}\,.$$

Hint: Apply the Calderón-Zygmund decomposition to the function $|f(x)|^p$ at the level 1.

Date: 12112016/RHS.