

**HARMONIC ANALYSIS**  
**2016**

7. HOMEWORK SHEET  
10.11.2016

Some review questions.

7.1. **Homework.** Let  $f$  be defined as  $f(x) = 10$  for  $x \in \mathbb{Q}$  and  $f(x) = 2^{-1}$  whenever  $x \in \mathbb{R} \setminus \mathbb{Q}$ . Find the supremum of  $f$ . Find the essential supremum of  $f$ , that is,  $\text{ess sup } f(x)$ .

7.2. **Homework.** Let  $D$  be a domain in  $\mathbb{R}^n$  with  $0 < |D| < \infty$  and let  $f$  be a measurable function defined in  $D$ . Show that

$$\|f\|_{L^\infty(D)} = \limsup_{p \rightarrow \infty} \left( \frac{1}{|D|} \int_D |f(x)|^p dx \right)^{1/p}.$$

7.3. **Homework.** Let  $f \in BMO(\mathbb{R}^n)$ ,  $h \in \mathbb{R}^n$ , and  $a > 0$ . Show that the translation  $\tau_h(f)$  and the dilation  $\delta_a(f)$  are in  $BMO(\mathbb{R}^n)$ .

7.4. **Homework.** Let  $f \in BMO(\mathbb{R}^n)$  and  $g \in BMO(\mathbb{R}^n)$ . Show that  $|f| \in BMO(\mathbb{R}^n)$ ,  $\max\{f, g\} \in BMO(\mathbb{R}^n)$ , and  $\min\{f, g\} \in BMO(\mathbb{R}^n)$ .

7.5. **Homework.** Show for the sharp maximal operator  $M^\#$ ,

$$\frac{1}{2} M^\# f(x) \leq \sup_Q \inf_{a \in \mathbb{R}} \frac{1}{|Q|} \int_Q |f(y) - a| dy \leq M^\# f(x)$$

where the supremum is taken over all cubes in  $\mathbb{R}^n$  containing  $x$ ,

$$M^\#(|f|)(x) \leq 2M^\# f(x),$$

$$M^\#(f + g)(x) \leq M^\# f(x) + M^\# g(x).$$