REAL-VARIABLE HARMONIC ANALYSIS I
2016

5. Homework sheet
13.10.2016

5.1. **Homework.** Let $0 < \alpha < n$ and $q = n/(n-\alpha)$. Let us write

$$(\mathcal{I}_\alpha f)(x) = \gamma(\alpha, n) \int_{\mathbb{R}^n} |x-y|^{\alpha-n} f(y)\, dy\,.$$

If $f \in L^1(\mathbb{R}^n)$, prove that there is a constant C such that

$$|\{I_\alpha f > \lambda\}| \le \frac{C}{\lambda^q} ||f||^q_{L^1(\mathbb{R}^n)}$$

for any $\lambda > 0$.

5.2. **Homework.** Let Q be a cube from a Whitney decomposition of a set Ω. Let$1 < \sigma < 5/4$ be given. Show that each point in Ω is contained in at most $(12)^n$ of the cubes σQ.

5.3. **Homework.** Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and let x be a Lebesgue point of f. Let $\mathcal{F}$ be a family of measurable subsets of $\mathbb{R}^n$ which are bounded and have positive measure. Suppose that there is a sequence (S_n), $S_n \in \mathcal{F}$, with $|S| \to 0$. Suppose that there is a constant $C > 0$ such that every $S \in \mathcal{F}$ is contained in a ball B_S centered at x such that $|S| \ge C|B_S|$. Prove that

$$\lim_{\substack{S\in\mathcal{F}\\ |S|\to 0}} \frac{1}{|S|} \int_S f(y)\, dy = f(x).$$

5.4. **Homework.** Let $f \in L^1(\mathbb{R}^n)$ and let $\alpha > 0$ be given. Let $\{Q_k\}$ be a family of cubes chosen from the Calderon-Zygmund decomposition of $\mathbb{R}^n$ and write $Q_k^* = 2Q_k$. Show that

$$\{x \in \mathbb{R}^n : M_2 f(x) > 7^n \alpha\} \subset \bigcup_k Q_k^*\,.$$

Here, $M_2 f$ is a Hardy-Littlewood maximal function for cubes $Q(x,r)$,

$$M_2 f(x) = \sup_{r>0} \frac{1}{|Q(x,r)|} \int_{Q(x,r)} |f(y)|\, dy.$$

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