REAL-VARIABLE HARMONIC ANALYSIS I
2016

4. HOMEWORK SHEET
6.10.2016

4.1. **Homework.** Let $1 < p < \infty$. Suppose that $f \in L^p(\mathbb{R}^n)$.
If $0 < \alpha p < n$ and $\delta > 0$, show that there is a constant $c(n, p, \alpha)$ so that

$$\int_{\mathbb{R}^n \setminus B_\delta(x)} \frac{|f(y)|}{|x-y|^{n-\alpha}}\, dy \le c(n,p,\alpha)\delta^{\alpha-(n/p)}||f||_{L^p}\,.$$

4.2. **Homework.** Let $1 < p < \infty$. Suppose that $f \in L^p(\mathbb{R}^n)$.
If $0 < \alpha p < n$ show that there is a constant $c(n, p, \alpha)$ so that

$$\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-\alpha}}\, dy \right)^{p^*} dx \le c(n,p,\alpha)||f||^{p^*}_{L^p(\mathbb{R}^n)}\,.$$

Here, $p^* = np/(n - \alpha p)$.
Lars-Inge Hedberg (around 1972) proved this inequality using the Hardy-Littlewood maximal inequality, and the method is called Hedberg's method.

4.3. **Homework.** Let $1 < p < n$. Suppose that $f \in C_0^\infty(\mathbb{R}^n)$. By using the previous results prove the Sobolev inequality: there exists a constant $c = c(n, p)$ so that

$$||f||_{L^{p^*}((\mathbb{R}^n)} \le c||\nabla f||_{L^p(\mathbb{R}^n)}\,.$$

4.4. **Homework.** Show that for cubes Q_1 and Q_2 from the Whitney decomposition of Ω with the property $\overline{Q_1} \cap \overline{Q_2} \neq \emptyset$ the inequalities

$$\frac{1}{4} \le \frac{\operatorname{diam}(Q_1)}{\operatorname{diam}(Q_2)} \le 4$$

holds.

4.5. **Homework.** Let Q be a cube from the Whitney decomposition of Ω. Show that there are at most $(12)^n$ cubes Q' from the Whitney decomposition such that

$$\overline{Q} \cap \overline{Q'} \neq \emptyset\,.$$

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