Department of Mathematics and Statistics
Geometric measure theory
Exercise 2
Solutions

1. **Problem.** *Let* $\mu\colon \mathcal{M} \to [0,+\infty]$ *be a measure defined on a* σ*-algebra* $\mathcal{M} \subset \mathcal{P}(X)$*. Define* $\mu^*\colon \mathcal{P}(X) \to [0,+\infty]$ *by setting*

$$\mu^*(A) = \inf\{\mu(B)\colon A \subset B \in \mathcal{M}\}.$$

(a) *Prove that* μ^* *is an outer measure in* X.
(b) *Prove that every* $E \in \mathcal{M}$ *is* μ^**-measurable and* $\mu^*|\mathcal{M} = \mu$.

Solution. Proof of (a): Clearly $\mu^*(\emptyset) = 0$. Let then

$$A \subset \bigcup_{i=1}^{\infty} A_i, \quad A, A_i \in \mathcal{P}(X).$$

Let $\epsilon > 0$. For each i choose $B_i \in \mathcal{M}$ such that $A_i \subset B_i$ and

$$\mu^*(A_i) \geq \mu(B_i) - \epsilon/2^i.$$

Then

$$A \subset \bigcup_{i=1}^{\infty} B_i \in \mathcal{M}$$

and

$$\begin{aligned}\mu^*(A) \leq \mu\left(\cup_{i=1}^{\infty} B_i\right) &\leq \sum_{i=1}^{\infty} \mu(B_i)\\ &\leq \sum_{i=1}^{\infty} \left(\mu^*(A_i) + \epsilon/2^i\right)\\ &\leq \sum_{i=1}^{\infty} \mu^*(A_i) + \epsilon.\end{aligned}$$

This holds for every $\epsilon > 0$, hence

$$\mu^*(A) \leq \sum_{i=1}^{\infty} \mu^*(A_i),$$

and therefore μ^* is an outer measure.

Proof of (b): If $E \in \mathcal{M}$, then $\mu(E) \leq \mu(B)$ for every $B \in \mathcal{M}$, with $E \subset B$. Hence

$$\mu^*(E) = \inf\{\mu(B)\colon E \subset B \in \mathcal{M}\} \geq \mu(E).$$

On the other hand, $\mu^*(E) \leq \mu(E)$, so $\mu^*(E) = \mu(E)$, and $\mu^*|\mathcal{M} = \mu$. We claim that every $E \in \mathcal{M}$ is μ^*-measurable. [Proof of the claim] Let $A \subset X$ be a test set and let $\epsilon > 0$. Choose $B \in \mathcal{M}$ such that $A \subset B$ and

$$\mu(B) - \epsilon \leq \mu^*(A) \leq \mu(B).$$

Then $A \cap E \subset B \cap E \in \mathcal{M}$ and $A \setminus E \subset B \setminus E \in \mathcal{M}$, and therefore

$$\begin{aligned}\mu^*(A \cap E) + \mu^*(A \setminus E) &\leq \mu(B \cap E) + \mu(B \setminus E)\\ &\overset{E \in \mathcal{M}}{=} \mu(B) \leq \mu^*(A) + \epsilon.\end{aligned}$$

This holds for every $\epsilon > 0$, so

$$\mu^*(A \cap E) + \mu^*(A \setminus E) \leq \mu^*(A),$$

and therefore E is μ^*-measurable.

2. **Problem.** *Let* m^* *be the Lebesgue outer measure in* $\mathbb{R}$, $A \subset \mathbb{R}$ *a non-Lebesgue measurable set,* $\tilde{\mu} = m^* \llcorner A$, *and*

$$\mu = \tilde{\mu}|\{E \subset \mathbb{R}\colon E \ \tilde{\mu}\text{-measurable}\}.$$

Prove that μ *is a Radon measure but not Borel regular.*

Solution. Write
$$\mathcal{M} = \{B \subset \mathbb{R} : B \text{ on } \tilde{\mu}\text{-mitallinen}\}.$$
If $B \in \mathrm{Leb}(\mathbb{R})$, then for every $E \subset \mathbb{R}$ we have
$$\begin{aligned}\tilde{\mu}(E) = m^*(E \cap A) &= m^*\big((E \cap A) \cap B\big) + m^*\big((E \cap A) \setminus B\big) \\ &= m^*\big((E \cap B) \cap A\big) + m^*\big((E \setminus B) \cap A\big) = \tilde{\mu}(E \cap B) + \tilde{\mu}(E \setminus B),\end{aligned}$$
and so $B \in \mathcal{M}$. Hence $\mathrm{Leb}(\mathbb{R}) \subset \mathcal{M}$ and, in particular, μ is a Borel measure. The measure μ is clearly locally finite, and therefore μ is a Rodon measure.

If $B \subset \mathbb{R} \setminus A$, then $\tilde{\mu}(B) = m^*(B \cap A) = 0$. Hence all subsets $B \subset \mathbb{R} \setminus A$ are $\tilde{\mu}$-measurable. In particular $\mathbb{R} \setminus A \in \mathcal{M}$. We claim that there exists no $B \in \mathrm{Bor}(\mathbb{R})$ such that $\mathbb{R} \setminus A \subset B$ and $\mu(\mathbb{R} \setminus A) = \mu(B)$. Suppose, on the contary, that such a set $B \in \mathrm{Bor}(\mathbb{R})$ exists. Then
$$A = (A \setminus B) \cup (A \cap B) = B^c \cup (A \cap B),$$
and therefore $A \cap B$ can not be Lebesgue measurable. In particular, $m^*(A \cap B) > 0$, and so
$$\mu(A^c) = 0 < m^*(A \cap B) = \mu(B).$$
Hence there does not exist a Borel set B such that $\mathbb{R} \setminus A \subset B$ and $\mu(A^c) = \mu(B)$. Therefore, μ is not Borel regular.

3. **Problem.** *Let (X, τ) be a topological space and $A \in \mathrm{Bor}(X)$. Equip A with the relative topology $\tau|A$. Prove that*
$$\mathrm{Bor}(A) = \mathrm{Bor}(X)|A := \{A \cap B \colon B \in \mathrm{Bor}(X)\}.$$

Solution. Inclusion $\subset$: If $U \in \tau$, then $U \in \mathrm{Bor}(X)$, and therefore $U \cap A \in \mathrm{Bor}(X)|A$. Hence $\tau|A \subset \mathrm{Bor}(X)|A$. It is easy to see that $\mathrm{Bor}(X)|A$ is a σ-algebra in A and since it contains $\tau|A$, we have $\mathrm{Bor}(A) \subset \mathrm{Bor}(X)|A$.

Inclusion $\supset$: Write $\mathcal{B} = \{B \in \mathrm{Bor}(X) : B \cap A \in \mathrm{Bor}(A)\}$. It suffices to prove that $\mathcal{B} = \mathrm{Bor}(X)$. Now $\tau \subset \mathcal{B}$ by the definition of the relative topology $\tau|A$. Hence it is enough to verify that $\mathcal{B}$ is a σ-algebra. This is straightforward to check and we omit it.

4. **Problem.** *Construct a σ-finite Borel measure $\mu \colon \mathrm{Bor}(\mathbb{R}) \to [0, +\infty]$ that is not a Radon measure.*

Solution. Define $w : \mathbb{R} \to \mathbb{R}$ by setting
$$w(x) = \begin{cases} 1/x, & \text{if } x \in (0,1), \\ 0 & \text{otherwise} \end{cases}$$
and define $\mu : \mathrm{Leb}(\mathbb{R}) \to [0, \infty]$,
$$\mu(E) = \int_E w(x)\, dm(x).$$
Then μ is a measure and, moreover, it is a Borel measure because $\mathrm{Bor}(\mathbb{R}) \subset \mathrm{Leb}(\mathbb{R})$. Now $\mu\big(\mathbb{R} \setminus (0,1)\big) = 0$ and
$$\mu\big((1/i, 1)\big) = \int_{(1/i,1)} w(x)\, dm(x) = \int_{1/i}^1 \frac{dx}{x} = \log i < \infty$$
for $i \geq 2$. It follows that μ is σ-finite because
$$\mathbb{R} = \big(\mathbb{R} \setminus (0,1)\big) \cup \bigcup_{i=2}^{\infty} (1/i, 1).$$
Hence μ is a σ-finite Borel measure. However, it is not a Radon measure because the measure of the compact set $[0,1]$ is
$$\mu([0,1]) = \int_{[0,1]} w(x)\, dm(x) = \int_0^1 \frac{dx}{x} = \infty.$$

5. **Problem.** *Let X be a separable metric space and $A_k \subset X,\ k \in \mathbb{N}$. Prove that*
$$\dim_{\mathcal{H}}\big(\bigcup_{k=1}^{\infty} A_k\big) = \sup_k \dim_{\mathcal{H}}(A_k).$$

Solution. Write $A = \bigcup_{k=1}^{\infty} A_k$.

If $B_1 \subset B_2 \subset X$, then $\mathcal{H}^s(B_1) \leq \mathcal{H}^s(B_2)$ for all $s \geq 0$, and so $\dim_{\mathcal{H}}(B_1) \leq \dim_{\mathcal{H}}(B_2)$. In particular, $\dim_{\mathcal{H}}(A_k) \leq \dim_{\mathcal{H}}(A)$ for all k, and therefore also $\sup_k \dim_{\mathcal{H}}(A_k) \leq \dim_{\mathcal{H}}(A)$.

If $\dim_{\mathcal{H}}(A) = 0$, then $\sup_k \dim_{\mathcal{H}}(A_k) = 0$ and we are done. Thus we may assume that $\dim_{\mathcal{H}}(A) > 0$. Let $0 < s < \dim_{\mathcal{H}}(A)$ so that $\mathcal{H}^s(A) = \infty$. Then

$$\infty = \mathcal{H}^s(A) \leq \sum_{k=1}^{\infty} \mathcal{H}^s(A_k)$$

and, in particular, there exists $m \in \mathbb{N}$ such that $\mathcal{H}^s(A_m) > 0$. Since $\mathcal{H}^s(A_m) > 0$, we have $\dim_{\mathcal{H}}(A_m) \geq s$, and therefore $\sup_k \dim_{\mathcal{H}}(A_k) \geq s$. This holds for all $0 < s < \dim_{\mathcal{H}}(A)$, hence $\sup_k \dim_{\mathcal{H}}(A_k) \geq \dim_{\mathcal{H}}(A)$.