Department of Mathematics and Statistics
Geometric measure theory
Exercise 1
Solutions

1. **Problem.** *Let $\mathcal{F} \subset \mathcal{P}(X)$ and*

$$\sigma(\mathcal{F}) = \bigcap \{\mathcal{M} \colon \mathcal{M} \text{ is a } \sigma\text{-algebra in } X,\ \mathcal{F} \subset \mathcal{M}\}.$$

Prove that $\sigma(\mathcal{F})$ is a σ-algebra.

Solution. Write

$$Y = \{\mathcal{M} : \mathcal{M} \text{ is a } \sigma\text{-algebra in } X,\ \mathcal{F} \subset \mathcal{M}\},$$

so that $\sigma(\mathcal{F}) = \bigcap Y$.

Since $\emptyset \in \mathcal{M}$ for all $\mathcal{M} \in Y$, we have $\emptyset \in \bigcap Y = \sigma(\mathcal{F})$.

Let $A \in \sigma(\mathcal{F})$. Then $A \in \mathcal{M}$ for all $\mathcal{M} \in Y$. Since every $\mathcal{M} \in Y$ is a σ-algebra, also $X \setminus A \in \mathcal{M}$ for each $\mathcal{M} \in Y$. Hence $X \setminus A \in \sigma(\mathcal{F})$.

Finally, assume that $A_1, A_2, \ldots \in \sigma(\mathcal{F})$. Then $A_1, A_2, \ldots \in \mathcal{M}$ for every $\mathcal{M} \in Y$. Since every $\mathcal{M}$ is a σ-algebra, also $\bigcup_i A_i \in \mathcal{M}$ for each $\mathcal{M} \in Y$. Hence $\bigcup_i A_i \in \bigcap Y = \sigma(\mathcal{F})$.

2. **Problem.** *Prove that every closed subset of a metric space is a $\mathcal{G}_\delta$ set and every open set (in a metric space) is an $\mathcal{F}_\sigma$ set.*

Solution. Let (X, d) be a metric space.

Let $U \subset X$ be open. We claim that

$$U = \bigcup_{i=1}^{\infty} F_i$$

for some closed sets F_i. We may assume that $U \neq X$ since otherwise U itself is closed. Define

$$F_i = \{x \in X \colon \operatorname{dist}(x, U^c) \geq 1/i\},\ i \in \mathbb{N}.$$

Since $U^c \neq \emptyset$, the function $x \mapsto \operatorname{dist}(x, U^c)$ is well-defined and continuous (1-Lipschitz). Hence each F_i is closed as a preimage of a closed set $[1/i, +\infty)$. For $x \in U$, $r := \operatorname{dist}(x, U^c) > 0$, and therefore $x \in F_i$ for $i > 1/r$. Hence $U \subset \bigcup_i F_i$. On the other hand, $F_i \subset U$ trivially, and so $U = \bigcup_i F_i$. Hence U is $\mathcal{F}_\sigma$ set.

Let then $F \subset X$ be closed. Then F^c is open, and therefore on

$$F^c = \bigcup_i F_i$$

for some closed $F_1, F_2, \ldots$. Hence

$$F = \Big(\bigcup_i F_i\Big)^c = \bigcap_i F_i^c$$

is a $\mathcal{G}_\delta$ set.

3. **Problem.** *Let $\tilde{\mu}$ be an outer measure in a metric space X such that every Borel set in X is $\tilde{\mu}$-measurable (i.e. $\tilde{\mu}$ is a Borel outer measure). Prove that $\tilde{\mu}$ is a metric outer measure.*

Solution. Let $A, B \subset X$ be such that $\operatorname{dist}(A, B) > 0$. We claim that $\tilde{\mu}(A \cup B) = \tilde{\mu}(A) + \tilde{\mu}(B)$. The closure $\bar{A}$ is closed, hence a Borel set, and therefore $\tilde{\mu}$-measurable. Hence for every test set $E \subset X$ we have

$$\tilde{\mu}(E) = \tilde{\mu}(E \cap \bar{A}) + \tilde{\mu}(E \setminus \bar{A}).$$

Using $E = A \cup B$ as a test set we get

$$\tilde{\mu}(A \cup B) = \tilde{\mu}\big((A \cup B) \cap \bar{A}\big) + \tilde{\mu}\big((A \cup B) \setminus \bar{A}\big) = \tilde{\mu}(A) + \tilde{\mu}(B)$$

since $A \subset \bar{A}$ and $B \subset (\bar{A})^c$.

4. **Problem.** *Let $\tilde{\mu}$ be a Borel regular outer measure in X and let $A \subset X$ be $\tilde{\mu}$-measurable such that $\mu(A) < \infty$. Prove that $\tilde{\mu} \llcorner A$ is Borel regular.*

Solution. Recall that $\tilde{\mu}\llcorner A$ is an outer measure that is defined as $(\tilde{\mu}\llcorner A)(B) = \tilde{\mu}(A \cap B)$.

First we claim that Borel sets are $\tilde{\mu}\llcorner A$-measurable. Let $B \subset X$ be Borel. Since $\tilde{\mu}$ is a Borel outer measure, B is $\tilde{\mu}$-measurable. Let $E \subset X$ be arbitrary and use $E \cap A$ as a test set (for $\tilde{\mu}$-measurable set B). We obtain

$$\begin{aligned}(\tilde{\mu}\llcorner A)(E) &= \tilde{\mu}(E \cap A)\\ &= \tilde{\mu}\big((E \cap A) \cap B\big) + \tilde{\mu}\big((E \cap A) \setminus B\big)\\ &= \tilde{\mu}\big((E \cap B) \cap A\big) + \tilde{\mu}\big((E \setminus B) \cap A\big)\\ &= (\tilde{\mu}\llcorner A)(E \cap B) + (\tilde{\mu}\llcorner A)(E \setminus B).\end{aligned}$$

Hence B is $\tilde{\mu}\llcorner A$-measurable, and so $\tilde{\mu}\llcorner A$ is a Borel outer measure.

Let then $E \subset X$ be arbitrary. We want to find a Borel set B such that $E \subset B$ and $(\tilde{\mu}\llcorner A)(E) = (\tilde{\mu}\llcorner A)(B)$. Since $\tilde{\mu}$ is Borel regular, there exits a Borel set B_1 such that $A \subset B_1$ and $\tilde{\mu}(A) = \tilde{\mu}(B_1)$. Similarly, there exits a Borel set B_2 such that $E \cap B_1 \subset B_2$ and $\tilde{\mu}(E \cap B_1) = \tilde{\mu}(B_2)$. We now choose

$$B = B_1^c \cup B_2.$$

Then B is a Borel set and

$$E = (E \cap B_1) \cup (E \setminus B_1) \subset B_2 \cup B_1^c = B.$$

Since $B_1 \setminus A$ and A are $\tilde{\mu}$-measurable and disjoint, we get

$$\tilde{\mu}(B_1 \setminus A) + \tilde{\mu}(A) = \tilde{\mu}\big((B_1 \setminus A) \cup A\big) = \tilde{\mu}(B_1) = \tilde{\mu}(A).$$

By assumption $\tilde{\mu}(A) < \infty$, and therefore we can conclude that $\tilde{\mu}(B_1 \setminus A) = 0$. We get

$$\begin{aligned}\tilde{\mu}(B \cap A) \geq \tilde{\mu}(E \cap A) &= \tilde{\mu}(E \cap A) + \tilde{\mu}(B_1 \setminus A)\\ &= \tilde{\mu}(E \cap A) + \tilde{\mu}\big(E \cap (B_1 \setminus A)\big)\\ &\geq \tilde{\mu}\big((E \cap A) \cup \big(E \cap (B_1 \setminus A)\big)\big)\\ &= \tilde{\mu}(E \cap B_1)\\ &= \tilde{\mu}(B_2)\\ &\geq \tilde{\mu}(B_2 \cap A)\\ &\overset{A \subset B_1}{=} \tilde{\mu}\big((B_1^c \cap A) \cup (B_2 \cap A)\big)\\ &= \tilde{\mu}(B \cap A).\end{aligned}$$

Now B is Borel, $E \subset B$, and $(\tilde{\mu}\llcorner A)(E) = (\tilde{\mu}\llcorner A)(B)$.

5. **Problem.** *Prove that in Exercise 4*

(a) *the assumption $\tilde{\mu}(A) < \infty$ can be replaced by an assumption $A \in \mathrm{Bor}(X)$, but*
(b) *in general, the assumption $\tilde{\mu}(A) < \infty$ is necessary. In other words, construct a topological space X, a Borel regular outer measure $\tilde{\mu}$ in X, an a $\tilde{\mu}$-measurable subset $A \subset X$ so that $\tilde{\mu}\llcorner A$ is not Borel regular.*

Solution. The assumption $\tilde{\mu}(A) < \infty$ can be replaced by an assumption $A \in \mathrm{Bor}(X)$ since we may choose $B_1 = A$ and the rest of the proof works (without assuming $\tilde{\mu}(A) < \infty$). To prove the claim (b), let $X = \mathbb{R}$ be equipped with the usual topology and let $\tilde{\mu}\colon \mathcal{P}(\mathbb{R}) \to [0, +\infty]$ be the counting (outer) measure. Then every subset of $\mathbb{R}$ is $\tilde{\mu}$-measurable, in particular, $\tilde{\mu}$ is a Borel outer measure. MOreover, $\tilde{\mu}$ is Borel regular. Indeed, if $A \subset \mathbb{R}$ is finite, then A is Borel (in fact, closed) and therefore its own Borel cover. Otherwise, A is infinite and we may choose $\mathbb{R}$ as its Borel cover because $\tilde{\mu}(A) = \infty = \tilde{\mu}(\mathbb{R})$. Fix a set $A \subset \mathbb{R}$ that is not Borel. Then necessarily, $\tilde{\mu}(A) = \infty$. Let $B \subset \mathbb{R}$ be a Borel set such that $\mathbb{R} \setminus A \subset B$. Then $A \cap B \neq \emptyset$, because otherwise $A = (A \cap B) \cup (A \setminus B) = A \setminus B$, and hence $\mathbb{R} \setminus A = \mathbb{R} \setminus (A \setminus B) = (\mathbb{R} \setminus A) \cup B = B$, and A would be Borel. Hence $\tilde{\mu}(A \cap B) > 0$, and so

$$(\tilde{\mu}\llcorner A)(\mathbb{R} \setminus A) = \tilde{\mu}(A \cap A^c) = 0 < \tilde{\mu}(A \cap B) = (\tilde{\mu}\llcorner A)(B).$$

We conclude that there exists no Borel sets $B \supset \mathbb{R} \setminus A$ such that $(\tilde{\mu}\llcorner A)(\mathbb{R} \setminus A) = (\tilde{\mu}\llcorner A)(B)$, and therefore $\tilde{\mu}\llcorner A$ is not Borel regular.

Another very short proof of (b): Let $X = \{0, 1\}$ be equipped with the trivial topology $\tau = \{\emptyset, X\}$. Then $\mathrm{Bor}(X, \tau\} = \tau$. Define an outer measure $\tilde{\mu}$ by setting

$$\tilde{\mu}(B) = \begin{cases} 0, & \text{if } B = \emptyset;\\ \infty, & \text{otherwise.}\end{cases}$$

Let $A = \{0\}$. Then $\tilde{\mu}\llcorner A$ is not Borel regular since $(\tilde{\mu}\llcorner A)(\{1\}) = \tilde{\mu}(\emptyset) = 0$ but $X = \{0, 1\}$ is the only Borel set containing $\{1\}$ and $(\tilde{\mu}\llcorner A)(\{1\}) = 0 \neq \infty = \tilde{\mu}(\{0\}) = (\tilde{\mu}\llcorner A)(X)$.