Department of Mathematics and Statistics
Geometric measure theory
Exercise 2
21.9.2016

1. Let $\mu\colon \mathcal{M} \to [0,+\infty]$ be a measure defined on a σ-algebra $\mathcal{M} \subset \mathcal{P}(X)$. Define $\mu^*\colon \mathcal{P}(X) \to [0,+\infty]$ by setting
$$\mu^*(A) = \inf\{\mu(B)\colon A \subset B \in \mathcal{M}\}.$$
   (a) Prove that μ^* is an outer measure in X.
   (b) Prove that every $E \in \mathcal{M}$ is μ^*-measurable and $\mu^*|\mathcal{M} = \mu$.

2. Let m^* be the Lebesgue outer measure in $\mathbb{R}$, $A \subset \mathbb{R}$ a non-Lebesgue measurable set, $\tilde{\mu} = m^* \llcorner A$, and
$$\mu = \tilde{\mu}|\{E \subset \mathbb{R}\colon E \ \tilde{\mu}\text{-measurable}\}.$$
Prove that μ is a Radon measure but not Borel regular.

3. Let (X,τ) be a topological space and $A \in \mathrm{Bor}(X)$. Equip A with the relative topology $\tau|A$. Prove that
$$\mathrm{Bor}(A) = \mathrm{Bor}(X)|A := \{A \cap B\colon B \in \mathrm{Bor}(X)\}.$$

4. Construct a σ-finite Borel measure $\mu\colon \mathrm{Bor}(\mathbb{R}) \to [0,+\infty]$ that is not a Radon measure.

5. Let X be a separable metric space and $A_k \subset X,\ k \in \mathbb{N}$. Prove that
$$\dim_{\mathcal{H}}\Big(\bigcup_{k=1}^{\infty} A_k\Big) = \sup_k \dim_{\mathcal{H}}(A_k).$$