Department of Mathematics and Statistics
Geometric measure theory
Exercise 1
14.9.2016

1. Let $\mathcal{F} \subset \mathcal{P}(X)$ and
$$\sigma(\mathcal{F}) = \bigcap\{\mathcal{M} \colon \mathcal{M} \text{ is a } \sigma\text{-algebra in } X,\, \mathcal{F} \subset \mathcal{M}\}.$$
Prove that $\sigma(\mathcal{F})$ is a σ-algebra.

2. Prove that every closed subset of a metric space is a $\mathcal{G}_\delta$ set and every open set (in a metric space) is an $\mathcal{F}_\sigma$ set.

3. Let $\tilde{\mu}$ be an outer measure in a metric space X such that every Borel set in X is $\tilde{\mu}$-measurable (i.e. $\tilde{\mu}$ is a Borel outer measure). Prove that $\tilde{\mu}$ is a metric outer measure.

4. Let $\tilde{\mu}$ be a Borel regular outer measure in X and let $A \subset X$ be $\tilde{\mu}$-measurable such that $\mu(A) < \infty$. Prove that $\tilde{\mu} \llcorner A$ is Borel regular.

5. Prove that in Exercise 4
   (a) the assumption $\tilde{\mu}(A) < \infty$ can be replaced by an assumption $A \in \operatorname{Bor}(X)$, but
   (b) in general, the assumption $\tilde{\mu}(A) < \infty$ is necessary. In other words, construct a topological space X, a Borel regular outer measure $\tilde{\mu}$ in X, an a $\tilde{\mu}$-measurable subset $A \subset X$ so that $\tilde{\mu} \llcorner A$ is not Borel regular.