1

Area Formula - Talk outline

Thm (Area formula) $f \in \mathrm{Lip}(\mathbb{R}^n \to \mathbb{R}^m)$, $n \le m$

(1) $$\int_A Jf(x)\, d\mathcal{L}^n(x) = \int_{\mathbb{R}^m} \mathcal{H}^0(A \cap f^{-1}(y))\, d\mathcal{H}^n(y), \quad A \in \mathcal{L}^n$$

(Jacobian integral) (Geometric Quantity) (Hausdorff integral)

$Jf(x)$: Jacobian of $f(x)$ i.e. $[\![Df]\!](x)$.

Spl case f is furthermore one-to-one.

(1') $$\int Jf\, d\mathcal{L}^n = \mathcal{H}^n(f(A))$$

Hausdorff area of set defined by Lipschitz function

(2) **(Change of variable formula)**

$$\int_{\mathbb{R}^n} g(x)\, Jf(x)\, dx = \int_{\mathbb{R}^m} \Big[\sum_{x \in f^{-1}(y)} g(x) \Big]\, d\mathcal{H}^n(y)$$

$g \in L^1(\mathbb{R}^n, \mathcal{L}^n)$.

(Spl case: $g(x)$ as prob. measure).

Some notation

$A \in \mathcal{L}^n(\mathbb{R}^m)$: A: $\mathcal{L}^n$-measurable set in $\mathbb{R}^m$.

When $m = n$, we just write $A \in \mathcal{L}^n$.

Similarly for $A \in \mathcal{H}^n(\mathbb{R}^m)$.

$f \in \mathcal{L}^n$ says f is $\mathcal{L}^n$-measurable function etc.

(2)

RKs. In area formula, arise measures $\mathcal{L}^n$, $\mathcal{H}^0$, $\mathcal{H}^n$.

1\. Lebesgue measure as a full-dimensional measure. Lebesgue measure as a product measure.
$\mathcal{L}^n$ lives on $\mathbb{R}^n$. $\mathcal{L}^n$ defined as a product measure.
$\mathcal{L}^m$, $m < n$, 'collapses' on $\mathbb{R}^n$. ~~Whereas~~
Lebesgue measure depends on ambient space.

vs Hausdorff measure independent of ambient space.
Hausdorff measure - 'intrinsic' measure.
$\mathcal{H}^s$, $0 \leq s \leq n$ for coexist in $\mathbb{R}^n$.
(and even interact)
(In $\mathbb{R}^n$, $\mathcal{H}^s(\cdot)$ for any $s > n$ is zero.)

2\. $Jf(x)$ - discussion of Jacobian.

$S \in Sym(n)$
$O \in O(n,m)$

$L : \mathbb{R}^n \to \mathbb{R}^m$ $\qquad L = O \circ S$

$[\![L]\!] = |\det S|$.

Scaling map (dilations).
(rotations).

Brief discussion -

$L^* L : \mathbb{R}^n \to \mathbb{R}^n$,
$L^* L$ is a non-negative definite, symm. matrix.

Some simple 'examples' to illustrate area formula.

E1 Length of curve. $\qquad f \in Lip(\mathbb{R} \xrightarrow{1\text{-}1} \mathbb{R}^m)$.

$$\mathcal{H}^1(C) = \int_a^b |\dot{f}|\, dt$$

E2 Surface area of graphs. $\qquad \mathcal{H}^n(G) = \int_U (1 + |Dg|^2)^{1/2} dx$.

$$Df = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \\ \frac{\partial g}{\partial x_1} & \cdots & \frac{\partial g}{\partial x_n} \end{bmatrix}_{(n+1)\times n}$$

(3)

Thm on (Weighted Jacobian & Area formula) CoV Formula - 1.

~~& Area formula for Weighted Jacobian~~

Thm Let $f \in Lip(\mathbb{R}^n \to \mathbb{R}^m)$, $n \le m$.

Then for $\forall g \in L^1(\mathbb{R}^n, \mathcal{L}^n)$,

$$\int g(x) Jf(x)\, dx = \int_{\mathbb{R}^m} \left[\sum_{x \in f^{-1}(y)} g(x) \right] d\mathcal{H}^n(y)$$

Pf case 1. $g \ge 0$,

$$g = \sum_{i=1}^{\infty} \frac{1}{i} \chi_{A_i}, \text{ for } \{A_i\}_{i=1}^{\infty}, A_i \in \mathcal{L}^n.$$

MCT implies

$$\int_{\mathbb{R}^n} g Jf\, dx = \sum_{i=1}^{\infty} \frac{1}{i} \int_{\mathbb{R}^n} \chi_{A_i} Jf\, dx$$

$$= \sum_{i=1}^{\infty} \frac{1}{i} \boxed{\int_{A_i} Jf\, dx}$$

[using Area formula]

$$= \sum_{i=1}^{\infty} \frac{1}{i} \int_{\mathbb{R}^m} \underbrace{\mathcal{H}^0(A_i \cap f^{-1}(y))}_{(= N(f|A_i, y))} d\mathcal{H}^n(y)$$

$$= \int_{\mathbb{R}^m} \sum_{i=1}^{\infty} \frac{1}{i} \sum_{x \in f^{-1}(y)} \chi_{A_i}(x)\, d\mathcal{H}^n(y)$$

$$= \int_{\mathbb{R}^m} \sum_{x \in f^{-1}(y)} \left(\sum_{i=1}^{\infty} \frac{1}{i} \chi_{A_i}(x) \right) d\mathcal{H}^n(y)$$

$$= \int_{\mathbb{R}^m} \left[\sum_{x \in f^{-1}(y)} g(x) \right] d\mathcal{H}^n(y).$$

2. Case: $g \in L^1(\mathbb{R}^n, \mathcal{L}^n)$, write $g = g^+ - g^-$ and apply case 1.

Structure of the Proof

Lemma 1 (Area formula holds for linear maps)

$L \in \mathrm{Lin}(\mathbb{R}^n \to \mathbb{R}^m)$, $n \le m$. Then

$$\mathcal{H}^n(L(A)) = [\![L]\!]\, \mathcal{L}^n(A), \quad \forall A \subset \mathbb{R}^n.$$

Proof (brief outline). ~~omit density part~~

. Jacobian arises as RN Derivative - brief mention.

Lemma 2 (Approximation properties of Hausdorff area integral)

$A \in \mathcal{L}^n$, $f \in \mathrm{Lip}(\mathbb{R}^n \to \mathbb{R}^m)$.

(i) $f(A) \in \mathcal{H}^n$. $\left(A \in \mathcal{L}^n \xrightarrow{f \in \mathrm{Lip}} f(A) \in \mathcal{H}^n\right)$

ii) the mapping $N : y \longmapsto \mathcal{H}^0(A \cap f^{-1}(y))$.

$N \in \mathcal{H}^n(\mathbb{R}^m)$. ($N$: multiplicity function)

(iii) $\displaystyle\int_{\mathbb{R}^n} \mathcal{H}^0(A \cap f^{-1}(y))\, d\mathcal{H}^n \le (\mathrm{Lip}(f))^n\, \mathcal{L}^n(A)$.

Rk (iii) is ~~intuitive~~ ~~intuitively~~ ($\mathrm{Lip}(f)$) gives the maximum stretch in any direction, ~~the~~ ~~n-diml Hausd~~ the n-diml Hausdorf area is within the factor $(\mathrm{Lip}(f))^n$. of ~~the~~ input Leb

(5)

(Lemma 3) (Lipschitz Linearization & Borel Partitions)

Let $t > 1$, $E \equiv \{x \mid Df(x) \text{ exists}, Jf(x) > 0\}$

Then there exists a ω-family $\{E_k\}_{k=1}^{\infty}$ of Borel sets of $\mathbb{R}^n$ s.t.

(i) $E = \bigcup_{k=1}^{\infty} E_k$

ii) $f_k := f|E_k$ is 1-1 $(k \in \mathbb{N})$.

iii) $\forall k \in \mathbb{N} \quad \exists T_k \in GL(n) \cap Sym(n)$

$$\mathrm{Lip}(f_k \circ T_k^{-1}) \le t, \quad \mathrm{Lip}(T_k \circ f_k^{-1}) \le t$$

[This says that $f_k \circ T_k^{-1}$ is almost an isometry.]

$$t^{-n} JT_k \le Jf_k \le t^n JT_k$$

Some Explanations from the proof.

$\frac{1}{t}+\epsilon \quad 1 \quad t-\epsilon$

C - countable dense subset of E.

S - Countable dense subset of $GL(n) \cap Sym(n)$. (matrices with rational entries).

$\forall c \in C, \forall T \in S \ \forall i \in \mathbb{N}$

$E(c, T, i) \equiv \{ ~~b \in B~~ \ b \in E \cap B(c, 1/i) \ni$

$$\left(\frac{1}{t}+\epsilon\right)|Tv| \le |Df(b)v| \le (t-\epsilon)|Tv|, \forall v \in \mathbb{R}^n.$$

[the set of all b for which D_b the relative error in ~~approximating~~ between $D_b f$ and T is between $(\frac{1}{t}+\epsilon)$ and $(t-\epsilon)$]

(ie T is a 'good' approximation for $D_b f$, for ~~$\forall b \in E \cap B(c$~~ all b in a nbd of c)

6

Lemma 1 (Area formula for linear maps)

$L \in \mathrm{Lin}(\mathbb{R}^n \to \mathbb{R}^m)$, $n \le m$. Then

$$\mathcal{H}^n(L(A)) = [\![L]\!]\, \mathcal{L}^n(A), \quad \forall A \subset \mathbb{R}^n.$$

Pf 1. $L = O \circ S$, $[\![L]\!] = |\det S| = JS$

2. If $[\![L]\!] = 0$, then $\dim S(\mathbb{R}^n) \le n-1$,
so $\dim L(\mathbb{R}^n) \le n-1$, so $\mathcal{H}^n(L(\mathbb{R}^n)) = 0$.

3. If $[\![L]\!] > 0$, then

$$\frac{\mathcal{H}^n(L(B(x,r)))}{\mathcal{L}^n(B(x,r))} = \frac{\mathcal{L}^n(O^* \circ L(B(x,r)))}{\mathcal{L}^n(B(x,r))} = \frac{\mathcal{L}^n(O^* \circ O \circ S(B(x,r)))}{\mathcal{L}^n(B(x,r))}$$

$$= \frac{\mathcal{L}^n(S(x,r))}{\mathcal{L}^n(B(x,r))} = \frac{\mathcal{L}^n(S(B,0,1))}{\alpha(n)}$$

$$= |\det S| = [\![L]\!]$$

4. $\nu(A) = \mathcal{H}^n(L(A))$, $\forall A \subset \mathbb{R}^n$.
ν is a Radon measure,
$\nu \ll \mathcal{L}^n$.

$$D_{\mathcal{L}^n}\nu(x) = \lim_{r \to 0} \frac{\nu(B(x,r))}{\mathcal{L}^n(B(x,r))} = [\![L]\!].$$

Thus for $\forall B \in \mathrm{Bor}(\mathbb{R}^n)$,

$$\mathcal{H}^n(L(B)) = [\![L]\!]\, \mathcal{L}^n(B).$$

Since ν and $\mathcal{L}^n$ are Radon measures,
the same formula holds for all sets $A \subset \mathbb{R}^n$

$\square$

(7) <u>Lemma 2</u> (<u>Approximation Properties of Hausdorff area integral</u>) (<u>Lemma 2</u>)

let $A \in \mathcal{L}^n$. Then

(i) $f(A) \in \mathcal{H}^n(\mathbb{R}^m)$

(ii) the mapping $N: y \longmapsto \mathcal{H}^0(A \cap f^{-1}(y))$ is $\mathcal{H}^n$-measurable on $\mathbb{R}^m$. ie $N \in \mathcal{H}^n(\mathbb{R}^m)$

(iii) $\int_{\mathbb{R}^m} \mathcal{H}^0(A \cap f^{-1}(y))\, d\mathcal{H}^n \leq (\mathrm{Lip}(f))^n \mathcal{L}^n(A)$.

<u>Rk</u> The mapping $y \longmapsto \mathcal{H}^0(A \cap f^{-1}(y))$ is called the <u>multiplicity function</u> (& Banach indicatrix) and is denoted by N.

<u>Pf</u> 1. WLOG, assume that A is bounded.

2. Then $\exists$ Compact sets $K_i \subset A$ s.t.

$$\mathcal{L}^n(K_i) \geq \mathcal{L}^n(A) - \frac{1}{i} \quad \forall i \in \mathbb{N})$$

As $\mathcal{L}^n(A) < \infty$ and $A \in \mathcal{L}^n$, $\mathcal{L}^n(A - K_i) < 1/i$.

As f is continuous, $f(K_i)$ is compact and thus $f(K_i) \in \mathcal{H}^n$ [[$\mathcal{H}^n$ is Borel (outer) measure]]

Hence $f\left(\bigcup_{i=1}^{\infty} K_i\right) = \bigcup_{i=1}^{\infty} f(K_i) \in \mathcal{H}^n$.

Also

$$\mathcal{H}^n\left(f(A) - f\left(\bigcup_{i=1}^{\infty} K_i\right)\right) \leq \mathcal{H}^n\left(f\left(A - \bigcup_{i=1}^{\infty} K_i\right)\right)$$

$$\leq (\mathrm{Lip}(f))^n \mathcal{L}^n\left(A - \bigcup_{i=1}^{\infty} K_i\right) = 0$$

so $(f(A) - f(\bigcup K_i)) \in \mathcal{H}^n$.

$f(A) \in \mathcal{H}^n$ as $f(A) = \underset{[[(\in \mathcal{H}^n)]]}{f(\bigcup K_i)} \cup \underset{[[(\in \mathcal{H}^n)]]}{(f(A) - f(\bigcup K_i))}$.

This proves (i).

3. Let $\mathcal{B}_k = \left\{ Q \mid Q = \bigtimes_{i=1}^{n} (a_i, b_i], \; a_i = \frac{c_i}{k}, b_i = \frac{c_i+1}{k}, c_i \in \mathbb{Z}, i \in \mathbb{N} \right\}$.

$$\mathbb{R}^n = \bigsqcup_{Q \in \mathcal{B}_k} Q$$

$$g_k \equiv \sum \chi_{f(A \cap Q)},$$

$g_k \in \mathcal{H}^n$ [[by (i)]]

and $g_k(y) = \#\{Q \mid Q \in \mathcal{B}_k \text{ s.t. } f^{-1}(y) \cap (A \cap Q) \neq \emptyset\}$.

Thus $g_k(y) \uparrow \mathcal{H}^0(A \cap f^{-1}(y))$ as $k \to \infty$.

for each $y \in \mathbb{R}^m$.

Thus the map $N : y \longmapsto \mathcal{H}^0(A \cap f^{-1}(y))$

is $\mathcal{H}^n$-measurable.

4. By MCT

$$\int_{\mathbb{R}^m} \mathcal{H}^0(A \cap f^{-1}(y))\, d\mathcal{H}^n = \lim_{k \to \infty} \int g_k \, d\mathcal{H}^n$$

$$= \lim_{k \to \infty} \sum \mathcal{H}^n(f(A \cap Q))$$

$$\leq \overline{\lim_{k \to \infty}} \sum (\mathrm{Lip}(f))^n \mathcal{L}^n(A \cap Q)$$

$$= (\mathrm{Lip}(f))^n \mathcal{L}^n(A)$$

$\square$.

4. Relabel the ω-collection $\{E(c,T,i) \mid c \in C, T \in \mathbf{S}, i \in \mathbb{N}\}$ as $\{E_k\}_{k=1}^{\infty}$. [[For any $b \in B$, a suitable T, c, i can be chosen so that $b \in E(c,T,i)$]].

Select any $b \in B$, write $Df(b) = O \circ S$ as above, and choose $T \in \mathbf{S}$ s.t.

$$\mathrm{Lip}(T \circ S^{-1}) \le \left(\frac{1}{t}+\epsilon\right)^{-1}, \quad \mathrm{Lip}(S \circ T^{-1}) \le t - \epsilon$$

Now select $i \in \mathbb{N}$, and $c \in C$ so that $|b-c| < 1/i$,

$$|f(a) - \mathcal{J}f(a)| \le \frac{\epsilon}{\mathrm{Lip}(T^{-1})} |a-b| \le c\,|T(a-b)|$$

for all $a \in B(b, 2/i)$.

Then $b \in E(c,T,i)$.

Stmt (i) is proved.

5. Next choose any set E_k, which is of the form $E(c,T,i)$ for some $c \in C$, $T \in \mathbf{S}$, $i \in \mathbb{N}$.

Let $T_k = T$.

According to $(***)$,

$$\frac{1}{t}|T_k(a-b)| \le |f(a)-f(b)| \le t\,|T_k(a-b)|,$$

for $\forall b \in E_k$, $a \in B(b, 2/i)$.

As $E_k \subset B(c, 1/i) \subset B(b, 2/i)$,
[[see Defn of $E(c,T,i)$ in 2]] [[comes from choice of $E(c,T,i)$ in 2]]

we have

$$(****) \quad \frac{1}{t}|T_k(a-b)| \le |f(a)-f(b)| \le t\,|T_k(a-b)|$$

for all $a, b \in E_k$; hence $f|E_k$ is 1-1.

6. Finally, $(****)$ implies

$$\mathrm{Lip}(f_k \circ T_k^{-1}) \le t, \quad \mathrm{Lip}(T_k \circ f_k^{-1}) \le t$$

which leads to the estimate

$$t^{-n} JT_k \le Jf_k \le t^n JT_k$$

[[Why?]] · $E(c, T, i)$ is a Borel set since Df is Borel measurable.

· From (*) and (**), follows the estimate

$$\frac{1}{t}\,|T(a-b)| \leq |f(a)-f(b)| \leq t\,|T(a-b)|$$

for $b \in E(c,T,i)$, $a \in B(b, 2/i)$

[[ie the stretch of function f is within a t-factor of the strech of function T]].

3. claim $b \in E(c, T, i)$ [[if b is in the (suff small) vicinity/nbd of c]],

$$\left(\frac{1}{t}+\epsilon\right)^n JT \leq Jf(b) \leq (t-\epsilon)^n\, JT$$

[[ie the volume expansion of f (at b in the vicinity of c) is (approx.) within t^n-factor of volume expansion of T,]].

~~By (*)~~ Proof of claim ~~$(\frac{1}{t}+\epsilon)|Tv|$~~

$$Df(b) = L = O \circ S,$$

$$Jf(b) = [[Df(b)]] = |\det S| = JS$$

By (*) $\left(\frac{1}{t}+\epsilon\right)|Tv| \leq |(O\circ S)v| = |Sv| \leq (t-\epsilon)|Tv|$ for $\forall v \in \mathbb{R}^n$

~~[[put~~ [[by putting $v = T^{-1}v$ we set]]

so $\left(\frac{1}{t}+\epsilon\right)|v| \leq |(S\circ T^{-1})v| \leq (t-v)|v|$, $v \in \mathbb{R}^n$

Thus $(S\circ T^{-1})\, B(0,1) \subset B(0, t-\epsilon)$

so $|\det(S\circ T^{-1})|\,\alpha(n) \leq \mathcal{L}^n(B(0,t-\epsilon))$

$$= \alpha(n)\,(t-\epsilon)^n$$

hence $|\det S| \leq (t-\epsilon)^n |\det T|$.

The proof of the other inequality is

$$\left(\frac{1}{t}+\epsilon\right)^n |\det T| \leq |\det S|$$

is similar.

(11) Proof, Lemma 3 (Lipschitz Linearizations & Borel Partitions)

Lemma 3. Let $t > 1$, $E \equiv \{x \mid Df(x) \text{ exists}, Jf(x) > 0\}$.

Then $\exists$ ω-family $\{E_k\}$: Bor$(\mathbb{R}^n)$ s.t.

(i) $B = \bigcup_{k=1}^{\infty} E_k$

(ii) $f_k := f|E_k$ is one-to-one ($k \in \mathbb{N}$)

(iii) $\exists\ \forall k \in \mathbb{N}\ \exists T_k \in GL(n) \cap Sym(n)$ s.t.

$\text{Lip}(f_k \circ T_k^{-1}) \leq t$, $\text{Lip}(T_k \circ f_k^{-1}) \leq t$ — both f_k and T_k are invertible

[This formulation carries the idea that on E_k; $f_k \circ T_k^{-1}$ is almost an isometry]

& $t^{-n} JT_k \leq Jf_k \leq t^n JT_k$.

[volume expansion of f_k is within a factor t^n of the volume expansion of T_k (either above or below)]

Proof (Given $t > 1$.)

1. Fix $\epsilon > 0$ so that $\frac{1}{t} + \epsilon \quad 1 \quad t - \epsilon$

2. C : a countable dense set of B

S : a countable dense set of $Sym(n) \cap GL(n)$ (matrices with rational entries).

2. Then for $\forall c \in C$, $\forall T \in S$ $\forall i \in \mathbb{N}$

$$E(c, T, i) \equiv \Big\{ b \in B(c, 1/i) \ \Big|$$

(*) $\left(\left(\frac{1}{t} + \epsilon\right)|Tv| \leq |\underset{(Df(b)\,v)}{D_b f(v)}| \leq (t - \epsilon)|Tv| \quad \forall v \in \mathbb{R}^n. \right)$

[relative error between T and $D_b f$ is between $\frac{1}{t} + \epsilon$ and $(t - \epsilon)$]

[IOW, T is a 'good' approx. for D_b for $b \in B(c, 1/i)$]

and

(**) $|f(a) - \mathcal{T}f(a)| \leq \epsilon |Ta - Tb|$, $a \in B(b, 2/i)$

[$\mathcal{T}f(x) = f(b) + Df(b)(x - a)$, first order Taylor expansion around b.]

[The relative error in linear approx. of f is small].

Lemma 3 Let $t > 1$ and $E \equiv \{x \mid Df(x) \text{ exists and } Jf(x) > 0\}$. Then there is a countable collection $\{E_k\}$ of Borel sets of $\mathbb{R}^n$, such that

(i) $B = \bigcup E_k$

(ii) $f_k \equiv f|E_k$ is one-to-one ~~for~~, $k \in \mathbb{N}$.

(iii) For $\forall k \in \mathbb{N}$ $\exists T_k \in \mathrm{Sym}(n) \cap GL(n)$, s.t.

$$\mathrm{Lip}(f_k \circ T_k^{-1}) \leq t \; ; \; \mathrm{Lip}(T_k \circ f_k^{-1}) \leq t.$$

[$f_k \circ T_k^{-1}$ is almost an isometry].

$$t^{-n} JT_k \leq Jf_k \leq t^n JT_k$$

[volume growth under f_k is within a factor t^n of volume growth under T_k.]

[Hence the name of the Lemma].

As T_k are nonsingular linear dilations that approximate f locally, we say

A Lipschitz map is locally ~~an nondegenerate~~ (essentially) invertible dilations. 'superlinear'

~~"Weierstrass-like" thm.~~ "Silu"

A Lipschitz map is locally linear ~~random~~ invertible dilation

(Weierstrass-like thm) locally superlinear

$\boxed{\text{Lipschitz maps are well approximable by locally invertible dilations}}$

Proof of Area Formula

Thm (Area Formula) $f \in \mathrm{Lip}(\mathbb{R}^n \to \mathbb{R}^m)$, $1 \le n \le m$. Then

$$\int_A Jf(x)\, d\mathcal{L}^n(x) = \int_{\mathbb{R}^m} \mathcal{H}^0(A \cap f^{-1}(y))\, d\mathcal{H}^n(y)$$

$A \in \mathcal{L}^n(\mathbb{R}^n) = \mathcal{L}^n$

Pf 1. In view of Rademacher thm, we may as well assume $Df(x)$ and $Jf(x)$ exist for all $x \in A$. [[Why?]]

2. Case 1. $A \subset \{Jf > 0\}$.

- Fix $t > 1$
- choose Borel sets $\{E_k\}_{k=1}^{\infty}$ as in Lemma 3; we may assume that the sets $\{E_k\}_{k=1}^{\infty}$ are disjoint
- Define $\mathcal{B}_k$ as in proof of Lemma 3.

$$\mathcal{B}_k \equiv \Big\{ Q \mid Q = \bigtimes_{i=1}^{n} (a_i, b_i],\quad a_i = \frac{c_i}{k},\ b_i = \frac{c_i+1}{k},\ c_i \in \mathbb{Z},\ i \in \mathbb{N} \Big\}.$$

$$\mathbb{R}^n = \bigcup_{Q \in \mathcal{B}_k} Q$$

- Set $F^i_j = E_j \cap Q_i \cap A$ $\quad (Q_i \in \mathcal{B}_k,\ j = 1, 2, \cdots)$
- The sets F^i_j are disjoint and $A = \bigsqcup_{i,j=1}^{\infty} F^i_j$

3. Claim #1.

$$\lim_{k\to\infty} \sum \mathcal{H}^n(f(F^i_j)) = \int \mathcal{H}^0(A \cap f^{-1}(y))\, d\mathcal{H}^n$$

Pf of Claim #1: Let $g_k \equiv \sum_{i,j=1}^{\infty} \chi_{f(F^i_j)}$,

$g_k(y)$ is # sets $\{F^i_j\}$ s.t. $F^i_j \cap f^{-1}(y) \neq \emptyset$.

$g_k(y) \uparrow \mathcal{H}^0(A \cap f^{-1}(y))$ as $k \to \infty$.

Apply MCT.

4. $$\mathcal{H}^n(f(F_j^i)) = \mathcal{H}^n(f_j \circ T_j^{-1} \circ T_j(F_j^i)) \le t^n \mathcal{L}^n(T_j(F_j^i)) \qquad (4.1)$$

original big map   'pieces'

[[Lemma 3.(iii)]]

and

$$\mathcal{L}^n(T_j(F_j^i)) = \mathcal{H}^n(T_j \circ f_j^{-1} \circ f(F_j^i)) \le t^n \mathcal{H}^n(f(F_j^i)) \qquad (4.2)$$

[[Lemma 3.(iii)]]

Thus

$$t^{-2n}\mathcal{H}^n(f(F_j^i)) \le t^{-n}\mathcal{L}^n(T_j(F_j^i))$$

$$= t^{-n} \cdot JT_j \cdot \mathcal{L}^n(F_j^i)$$

$$[[\, t^{-n} JT_k \le Jf_k \le t^n JT_k \,]] \quad \le \int_{F_j^i} Jf\, dx$$

$$[[\, Jf_k \le t^n JT_k \,]] \quad \le t^n \underbrace{JT_j \mathcal{L}^n(F_j^i)}_{=}$$

$$= t^n \mathcal{L}^n(T_j(F_j^i)) \quad [[\text{Lemma 1}]],$$

$$[[\, t^{-n} JT_k \le Jf_k \,]] \quad \le t^{2n}\mathcal{H}^n(f(F_j^i))$$

we get

ie $$t^{-2n}\mathcal{H}^n(f(F_j^i)) \le \int Jf\, df \le t^{2n}\mathcal{H}^n(f(F_j^i))$$

Now sum on i and j:

$$t^{-2n}\sum_{i,j=1}^{\infty}\mathcal{H}^n(f(F_j^i)) \le \int Jf\, df \le t^{2n}\sum_{i,j=1}^{\infty}\mathcal{H}^n(f(F_j^i))$$

Now let $k \to \infty$ and use claim 1:

$$t^{-2n}\int \mathcal{H}^0(A \cap f^{-1}(y))\, d\mathcal{H}^n \le \int Jf \le t^{2n}\int \mathcal{H}^0(A \cap f^{-1}(y))\, d\mathcal{H}^n$$

Finally, let $t \to 1+$.

$\square$.

5. **Case 2.** $A \subset \{Jf = 0\}$.

Fix $0 < \epsilon \leq 1$.

Idea here is to reduce this case 2 to case 1, by the following artifice of extension map.

- $g : \mathbb{R}^n \longrightarrow \mathbb{R}^m \times \mathbb{R}^n$

  $x \longmapsto (fx, \epsilon x)$

- $p : \mathbb{R}^m \times \mathbb{R}^n \longrightarrow \mathbb{R}^m$

  $(y, z) \longmapsto y$

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "g"] \arrow[dr, dashed, "f"'] & \mathbb{R}^m \times \mathbb{R}^n \arrow[d, "p"] \\ & \mathbb{R}^m \end{tikzcd}$$

6. **Claim #2**: There exists a constant C such that

$$0 < Jg(x) < C\epsilon$$

(omit) **Pf** $Dg(x) = \begin{pmatrix} Df(x) \\ \epsilon I \end{pmatrix}_{(n+m) \times n}$.

$Jf(x)^2$ equals sum of squares of $(n \times n)$-subdeterminants of $Df(x)$ according to Binet-Cauchy formula, we get

$$Jg(x)^2 = \text{sum of squares of } (n \times n) \text{ subdeterminants of } Dg(x) \geq \epsilon^{2n} > 0.$$

Furthermore, since $|Df| \leq \mathrm{Lip}(f) < \infty$, we use Cauchy-Binet to get

$$Jg(x)^2 = Jf(x)^2 + \left\{ \begin{array}{l} \text{sum of squares} \\ \text{of terms each} \\ \text{involving at least one } \epsilon \end{array} \right\} \leq C\epsilon^2.$$

For each $x \in A$.

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7. Since $p : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$ is a projection, ~~we can~~ (ie g is an "extension" of f), using case 1 above, we get

$$\mathcal{H}^n(f(A)) \le \mathcal{H}^n(g(A)) \qquad [\mathbb{P}]$$

$$\le \int_{\mathbb{R}^{n+m}} \mathcal{H}^0(A \cap g^{-1}\{y,z\})\, d\mathcal{H}^n(y,z)$$

$$= \int_A Jg(x)\, dx. \qquad \text{[area formula from Case 1]}$$

$$\le \epsilon C\, \mathcal{L}^n(A) \qquad \text{[using } 0 < Jg(x) \le C\epsilon\text{]}$$

let $\epsilon \to 0$ to conclude, $\mathcal{H}^n(f(A)) = 0$,

and thus

$$\int_{\mathbb{R}^n} \mathcal{H}^n(A \cap f^{-1}(y))\, d\mathcal{H}^n = 0$$

Since $\operatorname{spt} \mathcal{H}^0(A \cap f^{-1}(y)) \subset f(A)$.

But then

$$\int_{\mathbb{R}^n} \mathcal{H}^0(A \cap f^{-1}(y))\, d\mathcal{H}^n = 0 = \int_A Jf\, dx.$$

8. General case: Write $A = A_1 \cup A_2$, where $A_1 \subset \{Jf > 0\}$, $A_2 \subset \{Jf = 0\}$.

and apply <u>Case 1</u> and <u>case 2</u>.