# Submanifolds of $\mathbb{R}^n$ and Mean Curvature

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# Overview

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# Manifolds

**Definition** A topological manifold is a second countable hausdorff space that is locally homeomorphic to euclidean unit ball $B^m \subset \mathbb{R}^m$ $B^m = \{x : |x| < 1\}$.

As we are working only in real space the definition reduces to:

**Definition** Real manifold $\mathcal{M}$ is a set of points of $\mathbb{R}^n$ with induced topology from $\mathbb{R}^n$ that is locally homeomorphic to $B^m$.

**Definition** Induced topology. Given a topological space $(\mathbb{R}^n, \tau)$ and a subset $S$ of $\mathbb{R}^n$ the induced topology is

$$\tau_S = \{S \cap U : U \in \tau\}$$

In our case $\tau$ is just the collection of open sets of $\mathbb{R}^n$

**Definition** Function $f : \mathbb{R}^n \to \mathbb{R}^m$ is called a homeomorphism if

1. $f$ is a bijection
2. $f$ is continuous
3. inverse function $f^{-1}$ is continuous

A function $f : \mathcal{M} \to \mathbb{R}^m$ is a local homeomorphism if for each $x \in \mathcal{M}$ there exists a open set $U$ such that the image $f(U)$ is open and the restriction $f|_U = f(\mathcal{M} \cap U)$ is a homeomorphism $f|_U : U \to f(U)$

**Example**

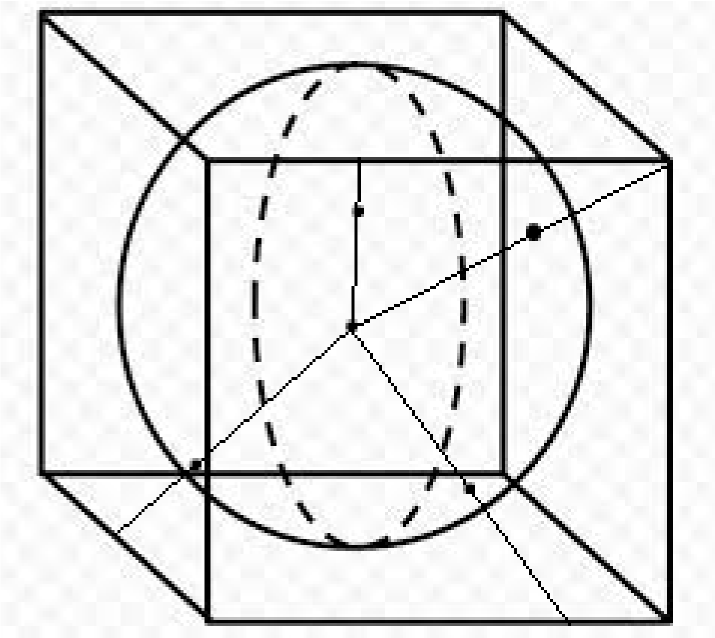
All open subsets of $R^n$ are manifolds. For all $p \in \mathcal{M}$ there exists by definition an open set $U$ such that $U \subset \mathcal{M}$ and $U$ is a open ball with center $p$ and radius $r$. Then choose as the homeomorphism $f(x) = \frac{x}{|r|} \circ (x - p) \circ id$.

**Definition**

Two manifolds $\mathcal{M}, \mathcal{N}$ are called homeomorphic if there exists a homeomorphism $f : \mathcal{M} \to \mathcal{N}$

**Example**

Sphere and a cube are homeomorphic.



# Diffeomorphisms

**Definition** A function $f : \mathbb{R}^n \to \mathbb{R}^n$ is called a diffeomorphism if it is a differentiable bijection and its inverse $f^{-1}$ is differentiable as well. If these functions are $r$ times continuosly differentiable then f is called a $C^r$-diffeomorphism.
In this text we generally assume that $f$ is atleast of class $C^3$

# Inverse function theorem

**Definition** Continuously differentiable function $f$ is invertible near a point $x \in \mathbb{R}^n$ if the Jacobian determinant at $x$ is non-zero

$$J_f(x) = \begin{bmatrix} \frac{\partial f_1(x)}{\partial x_1} & \dots & \frac{\partial f_1(x)}{\partial x_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial f_n(x)}{\partial x_1} & \dots & \frac{\partial f_n(x)}{\partial x_n} \end{bmatrix}$$

Converse is also true IF we assume that $f^{-1}$ is also continuously differentiable.

Theorem: If continuously differentiable function $f$ is invertible, and the function $f^{-1}$ is also continuously differentiable, then jacobian matrix $J_f(x)$ is inverible.
Proof:
By chain rule we can derive the identity for functions $G$ and $H$ which have total derivatives at $H(x)$ and $x$ respectively:

$$J_{G\circ H}(x) = J_G(H(x))J_H(x)$$

Then let $H = f$ and $G = f^{-1}$ and by noting that the jacobian of identity map is just $I$ the identity matrix we have:

$$I = J_{f^{-1}}(f(x))J_f(x)$$

So

$$[J_f(x)]^{-1} = J_{f^{-1}}(x)$$

# Spectral theorem

**Definition** Let $\langle ., . \rangle$ denote the standard inner product on $\mathbb{R}^n$. A matrix $A$ is then self adjoint (or symmetric) if $\langle Ax, y \rangle = \langle x, Ay \rangle$ for all $x, y \in \mathbb{R}^n$

**Definition** Spectral theorem for real and finite vector spaces: if matrix $A$ is self adjoint then it has orthonormal eigenbasis with real eigenvalues.

**Definition** A subset $M$ of $\mathbb{R}^{n+k}$ is a $n$-dimensional submanifold if for every point y in $M$ there exists open subsets $V, U$ of $\mathbb{R}^{n+k}$ and a diffeomorphism $\phi : U \to V$ such that $\phi(y) = 0$ and $\phi(M \cap U) \to W = V \cap \mathbb{R}^n$.

In particular we have local representation:

$$\psi = \phi^{-1}|W$$

$$\psi : W \to \mathbb{R}^{n+k}$$

So that $\psi(w) = \psi_1(w_1, ..., w_n), ..., \psi_{n+k}(w_1, ..., w_n)$ Note then that $\frac{\partial\psi(w)}{\partial w_1}, \frac{\partial\psi(w)}{\partial w_2}, ..., \frac{\partial\psi(w)}{\partial w_n}$ are linearly independent vectors of $\mathbb{R}^{n+k}$. This follows from the inverse function theorem.

We have the Jacobian matrix $J =$

$$\begin{bmatrix} \frac{\partial \psi_1(w)}{\partial w_1} & \cdots & \frac{\partial \psi_1(w)}{\partial w_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial \psi_n(w)}{\partial w_1} & \cdots & \frac{\partial \psi_n(w)}{\partial w_n} \\ \frac{\partial \psi_{n+1}(w)}{\partial w_1} & \cdots & \frac{\partial \psi_{n+1}(w)}{\partial w_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial \psi_{n+m}(w)}{\partial w_1} & \cdots & \frac{\partial \psi_{n+m}(w)}{\partial w_n} \end{bmatrix}$$

But by inverse function theorem since $\psi$ is invertible, we have that submatrix of J:

$$\begin{bmatrix} \frac{\partial \psi_1(w)}{\partial w_1} & \cdots & \frac{\partial \psi_1(w)}{\partial w_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial \psi_n(w)}{\partial w_1} & \cdots & \frac{\partial \psi_n(w)}{\partial w_n} \end{bmatrix}$$

Is invertible and by such its columns are linearly independent. Hence it follows that the columns of Matrix $J$ are also independent.
Let's then define the tangent plane $T_w\psi$ as the set spanned by the partial derivates $\frac{\partial\psi(w)}{\partial w_1}, \frac{\partial\psi(w)}{\partial w_2}, ..., \frac{\partial\psi(w)}{\partial w_n}$. To simplify notation we will from now on denote these vectors as $\psi(w)_{w_n} = \frac{\partial\psi(w)}{\partial w_n}$.
Notice how the dimension of the tangent space is the same as the dimension of the submanifold.

We imagine that the tangent plane is attached to the point $\psi(w)$. A vector $V(w) \in \mathbb{R}^{n+k}$ also attached to point $\psi(w)$ is called a tangent vector if $V(w) \in T_w\psi$ and normal if $V(w) \in T_w^{\perp}$

# Hypersurfaces

For the rest of this presentation we will be dealing only with submanifolds of dimension n in $\mathbb{R}^{n+1}$. The reason is that we can then easily define the normal vector $N(w)$ to the surface as one orthogonal to all tangent vectors $T_w$. (Gram-Schmidt)
Clearly vector $V(w)$ is tangent if and only if it can be expressed in the form $V(w) = \sum_{t=1}^{n} V(w)^t \psi(w)_{w_t}$ and it is normal if and only if it is of the form $V(w) = \lambda(w) N(w)$

Consider then the derivates of the normal vector $N(w)$. Since $\langle N(w), N(w)\rangle = 1$, we have:

$$\langle N(w)_{w_n}, N(w)\rangle = 0 \quad (1)$$

Where $N(w)_{w_n}$ denotes the derivate of $N(w)$ with respect to $w_n$. Note then that the vectors $N(w)_{w_n}$ are tangential, so we can define a mapping called the shape operator as:

$$S(w) : T_w\psi \to T_w\psi$$

which maps vector

$$V = \sum_{t=1}^{n} V(w)^t \psi(w)_{w_t}$$

onto vector

$$S(w)V = -\nabla_V N(w) = -\sum_{t=1}^{n} V(w)^t N_{w_t}$$

**Example 1.**

Let M be a plane:

$$ax + by + cy = d$$

Then the normal unit vector $N(w)$ is just the gradient $\frac{(a,b,c)}{|(a,b,c)|}$ But this is constant, so all $N(w)_{w_t}$ are zero so $S(w)V = 0$ for all $V \in T_t\psi$.

**Example 2.**

Let $M$ be sphere $S_r^2$. Consider then a curve $\omega : [-\epsilon, \epsilon] \to M$ on $M$ And let the tangent vector $V$ be $\frac{d\omega}{dt}(t)|_0$, and the unit normal is just $\frac{x}{|x|}$ at $x \in M$. Then,

$$-\nabla_V N(w)|_0 = -\frac{d}{dt}\frac{\omega(t)}{|\omega(t)|}|_0 = -\frac{1}{|\omega(t)|}\frac{d\omega}{dt}|_0 = -\frac{1}{|r|}V$$

Shape operator is selfadjoint linear mapping.
Proof: Choose arbitary tangent vectors $V = \sum_{t=1}^{n} V(w)^t \psi(w)_{w_t}$ and $W = \sum_{t=1}^{n} W(w)^t \psi(w)_{w_t}$. Then because

$$\langle N(w), \psi(w)_{w_a} \rangle = 0 \quad (2)$$

we have that

$$\langle N(w)_{w_b}, \psi(w)_{w_a} \rangle + \langle N(w), \psi(w)_{w_a w_b} \rangle = 0 \quad (3)$$

And

$$\langle N(w), \psi(w)_{w_b} \rangle = 0$$

so

$$\langle N(w)_{w_a}, \psi(w)_{w_b} \rangle + \langle N(w), \psi(w)_{w_b w_a} \rangle = 0$$

It then follows that

$$\langle N(w)_{w_a}, \psi(w)_{w_b}\rangle = \langle N(w)_{w_b}, \psi(w)_{w_a}\rangle \quad (4)$$

Therefore:

$\langle SV(w), W(w)\rangle = -\langle \sum_{t=1}^{n} V(w)^t N_{w_t}, \sum_{s=1}^{n} W(w)^s \psi(w)_{w_s}\rangle$

$$= -\sum_{t=1}^{n}\sum_{s=1}^{n} W(w)^s V(w)^t \langle N_{w_t}, \psi(w)_{w_s}\rangle$$

$$= -\sum_{t=1}^{n}\sum_{s=1}^{n} W(w)^s V(w)^t \langle \psi_{w_t}, N(w)_{w_s}\rangle$$

$$= -\langle \sum_{t=1}^{n} V(w)^t \psi_{w_t}, \sum_{s=1}^{n} W(w)^s N(w)_{w_s}\rangle$$

$= \langle V(w), SW(w)\rangle$

We may then define on $T_w\psi$ three symmetric bilinear forms:

$$\mathrm{I}(V(w), W(w)) = \langle V(w), W(w) \rangle,$$

$$\mathrm{II}(V(w), W(w)) = \langle SV(w), W(w) \rangle,$$

$$\mathrm{III}(V(w), W(w)) = \langle SV(w), SW(w) \rangle$$

For all $V(w), W(w) \in T_w\psi$, with their corresponding quadratic forms: $\mathrm{I}(V(w)) = |V|^2, \quad \mathrm{II}(V(w), V(w)) = \langle SV(w), V(w) \rangle,$ $\mathrm{III}(V(w), V(w)) = |SV(w)|^2$
Called first, second and third fundamental form.

Geometric meaning of second fundamental form is as follows. Consider an arbitary $C^3$-curve $\omega$ in $W$ which starts at 0, for example:

$$\omega : [0, \epsilon] \to W, \quad \omega(0) = 0, \quad \omega(t) = (\omega^1(t), \omega^2(t))$$

Then $c = \psi \circ \omega$ is a $C^3$ curve on surface $M$ with intial point $c(0) = \psi(0) = y$ and intial velocity

$$\dot{c}(0) = \sum_{k=1}^{n} \psi_{w_k}(\omega(0))\dot{\omega}^k(0) \in T_y\psi$$

Notice that, by definition:

$$|\dot{c}(0)|^2 = \mathrm{I}(\dot{c}(0))$$

Lets then assume that $t$ is the parameter of arc lenght $s$ of the curve $c$. So that we have $|\dot{c}(s)| = 1$. Moreover,

$$t(s) = \dot{c}(s)$$

is the unit tangent vector of the curve c and,

$$\kappa(s) = |\dot{t}(s)|$$

its curvature. For $\kappa(s) \neq 0$, the principal normal $n(s)$ is uniquely defined by the equation

$$\dot{t}(s) = \kappa(s) n(s)$$

Then from

$$t(s) = \dot{c}(s) = \sum_{k=1}^{n} \psi_{w_k}(\omega(s))\dot{\omega}^k(s)$$

we get

$$\dot{t}(s) = \sum_{k=1}^{n}\sum_{j=1}^{n} \psi_{w_k w_j}(\omega(s))\dot{\omega}^k(s)\dot{\omega}^j(s) + \sum_{k=1}^{n} \psi_{w_k}(\omega(s))\ddot{\omega}^k(s)$$

By taking the scalar product with $\dot{t}(0)$ with $N(w)$ where $N(w)$ is a unit normal to the surface at $c(0)$ we get:

$$\langle N(w), \dot{t}(0)\rangle = \sum_{k=1}^{n}\sum_{j=1}^{n} \langle N(w), \psi_{w_k w_j}\rangle \dot{\omega}^k(0)\dot{\omega}^j(0)$$

Remember then (3.)

$$\langle N(w)_{w_b}, \psi(w)_{w_a}\rangle + \langle N(w), \psi(w)_{w_a w_b}\rangle = 0$$

By using (3.) we get:

$$\langle N(w), \dot{t}(0)\rangle = -\sum_{k=1}^{n}\sum_{j=1}^{n}\langle N(w)_{w_j}, \psi_{w_k}\rangle \dot{\omega}^k(0)\dot{\omega}^j(0)$$

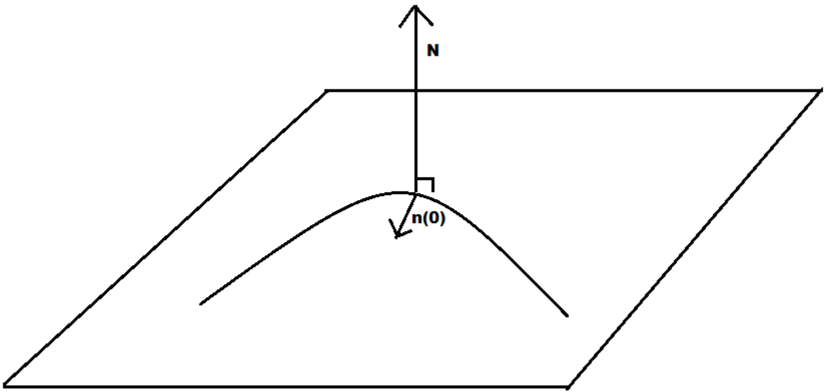
$$= \langle S\dot{c}(0), \dot{c}(0)\rangle$$

So

$$\langle N(w), \dot{t}(0)\rangle = \kappa(0)\langle N(w), n(0)\rangle = \langle S\dot{c}(0), \dot{c}(0)\rangle$$

Especially this means that

$$\kappa(0)\langle N(w), n(0)\rangle = \mathrm{II}(\dot{c}(0))$$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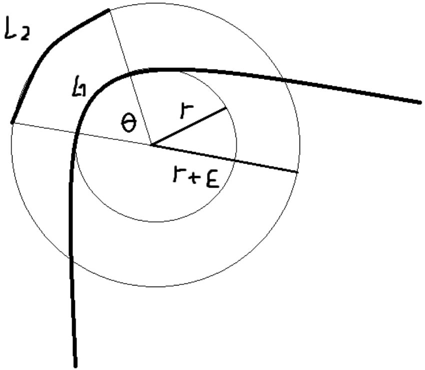
n(0)

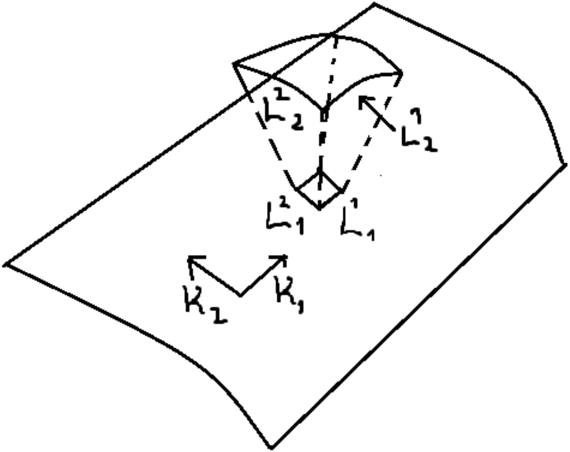
**Definition** Principal curvatures are the eigenvalues of the shape operator and principal directions are the eigenvectors of the shape operator.
Notice that by spectral theorem we have that principal curvatures are real and that principal directions are orthonormal eigenvectors of the shape operator.

**Definition** Mean curvature is defined as $\frac{1}{n}\sum_{k=1}^{n}\kappa_k$ Where $\kappa_k$ are the principal curvatures.
Note that this is the trace of of $S$ the shape operator.
What does this mean geometrically?




Consider the factor of change of lenght $L$ when we raise a segment of the curve in direction of normal. We denote this by $N_\epsilon(x)$ This is clearly

$$\frac{L_2}{L_1} = \frac{\theta(r+\epsilon)}{\theta r} = 1 + \frac{\epsilon}{r} = 1 + \epsilon\kappa$$




Similarly the factor of change of area is $(1 + \epsilon\kappa_1)(1 + \epsilon\kappa_2)$

And similarly generally for higher dimensional hypersurfaces the factor of change of volume is $\prod_{i=1}^{n}(1+\epsilon\kappa_i)$. Consider then a hypersurface $M$ in $\mathbb{R}^{n+1}$. Then

$$\mathcal{H}^n(N_\epsilon(M)) = \int_M \prod_{i=1}^{n}(1+\epsilon\kappa_i)d\mathcal{H}^n$$

$$= \int_M \mathcal{H}^n + \epsilon \int_M \sum_{i=1}^{n} \kappa_i d\mathcal{H}^n + O(\epsilon^2)$$

So when mean curvature is zero then the first "variation" is zero. Thus surfaces of zero mean curvature can be thought as minimal surfaces.