Geometric measure theory
Presentation: Monotonicity formula and isoperimetric inequality

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Throughout this presentation we assume that $V \in V_k(\mathbb{R}^n)$ and $\|\delta V\|$ is a Radon measure (this follows if we assume that V has locally bounded first variation).

We have shown that in this case there exists $\|\delta V\|$-measurable function $\eta : \mathbb{R}^n \to S^{n-1}$ for which

$$\delta V(X) = \int \langle X(x), \eta(x) \rangle \, d\|\delta V\|(x) \quad \forall X \in C_0^1(\mathbb{R}^n, \mathbb{R}^n)$$

Recall also that

$$\delta V(X) = \int \operatorname{div}_S X(x) \, dV(x,S) \quad \forall X \in C_0^1(\mathbb{R}^n, \mathbb{R}^n)$$

(This could be taken as the definition of δV.)

The density of V at $a \in \mathbb{R}^n$ is $d(V,a) = \lim_{r \to 0} d(V,a,r)$ when the limit exists. Here

$$d(V,a,r) = \frac{\|V\|(B(a,r))}{\alpha(k) r^k}$$

We will use the following auxiliary functions defined for each $a \in \mathbb{R}^n$

$A(a,\cdot) : [0,\infty) \to \mathbb{R}^+$, $\quad A(a,r) = \|V\|(B(a,r))$

$B(a,\cdot) : [0,\infty) \to \mathbb{R}^+$, $\quad B(a,r) = \displaystyle\int_{\{(x,S) : x \in B(a,r)\}} \left| S\left(\frac{x-a}{|x-a|}\right)\right|^2 dV(x,S)$

$C(a,\cdot) : [0,\infty) \to \mathbb{R}^+$, $\quad C(a,r) = \displaystyle\int_{\{(x,S) : x \in B(a,r)\}} \left| S^\perp\left(\frac{x-a}{|x-a|}\right)\right|^2 dV(x,S)$

$D(a,\cdot) : [0,\infty) \to \mathbb{R}$, $\quad D(a,r) = \displaystyle\int_{B(a,r)} \langle x-a, \eta(x) \rangle \, d\|\delta V\|(x)$

Here $S(\cdot) : \mathbb{R}^n \to S$ is the projection onto S and η is as before.

Lemma. (i) $A(a,r) = B(a,r) + C(a,r) \quad \forall a \in \mathbb{R}^n, \forall r \geq 0$

(ii) $D(a,r) + r B'(a,r) = k A(a,r) \quad \forall a \in \mathbb{R}^n, \forall r \geq 0$

(Differentiation w.r.t. variable r)

Proof: (i) Using the Pythagorean theorem we find

$$B(a,r) + C(a,r) = \int_{\{(x,S) : x \in B(a,r)\}} \left|S\left(\tfrac{x-a}{|x-a|}\right)\right|^2 + \left|S^\perp\left(\tfrac{x-a}{|x-a|}\right)\right|^2 dV(x,S)$$

$$= \int_{\{(x,S) : x \in B(a,r)\}} \underbrace{\left|\tfrac{x-a}{|x-a|}\right|}_{\equiv 1} dV(x,S) = \|V\|(B(a,r))$$

$$= A(a,r) \qquad \forall a \in \mathbb{R}^n \, \forall r \geq 0$$

(ii) Let $r \geq 0$ and $a \in \mathbb{R}^n$.

Let $\varepsilon > 0$. Choose a smooth cut-off function $\phi_\varepsilon : [0,\infty) \to \mathbb{R}$ for which $\phi_\varepsilon(t) = 1$ if $0 \leq t < r - \varepsilon$ and $\phi_\varepsilon(t) = 0$ if $t \geq r$. In addition we may assume $\phi_\varepsilon'(t) \leq 0$ for all $t \in [0,\infty)$. Define $X_\varepsilon : \mathbb{R}^n \to \mathbb{R}^n$ by setting

$$X_\varepsilon(x) = \phi_\varepsilon(|x-a|)(x-a) \quad \forall x \in \mathbb{R}^n$$

Then notice that (here $(\tau_i^S)_{i=1}^k$ is an orthonormal basis of S)

$$\delta V(X_\varepsilon) = \int \operatorname{div}_S \phi_\varepsilon(|x-a|)(x-a)\, dV(x,S)$$

$$= \int \sum_{i=1}^k \left[\phi_\varepsilon'(|x-a|) \frac{\langle x-a, \tau_i^S \rangle}{|x-a|} \langle x-a, \tau_i^S \rangle + \phi_\varepsilon(|x-a|) \overbrace{\langle \tau_i^S, \tau_i^S \rangle}^{=1}\right] dV(x,S)$$

$$= \int \phi_\varepsilon'(|x-a|)|x-a| \left|S\left(\tfrac{x-a}{|x-a|}\right)\right|^2 dV(x,S) + \int k \cdot \phi_\varepsilon(|x-a|)\, dV(x,S)$$

As $\varepsilon \to 0$ we notice that

$$\delta V(X_\varepsilon) = \int_{\mathbb{R}^n} \phi_\varepsilon(|x-a|) \langle x-a, \eta(x) \rangle \, d\|\delta V\| \longrightarrow \int_{B(a,r)} \langle x-a, \eta(x) \rangle d\|\delta V\| = D(a,r)$$

$$\int k \phi_\varepsilon(|x-a|)\, dV(x,S) \longrightarrow \int_{\{(x,S) : x \in B(a,r)\}} k \, dV(x,S) = k A(a,r)$$

$$\int \phi_\varepsilon'(|x-a|)|x-a| \left|S\left(\tfrac{x-a}{|x-a|}\right)\right|^2 dV(x,S) \longrightarrow -\int_{\{(x,S) : x \in \partial B(a,r)\}} \overbrace{|x-a|}^{\equiv r} \left|S\left(\tfrac{x-a}{|x-a|}\right)\right|^2 dV(x,S)$$

$$= -r B'(a,r)$$

So $D(a,r) = -rB'(a,r) + kA(a,r)$. This proves the claim.

$\square$

Akseli Haarala

Theorem. (Monotonicity formula) If $V \in \mathbf{V}_k(\mathbb{R}^n)$ is stationary i.e. $\delta V = 0$, then $r \longmapsto d(V, a, r)$ is not decreasing for every $a \in \mathbb{R}^n$.

Proof: Let $a \in \mathbb{R}^n$. Using the previous lemma we notice that

$$\begin{aligned}(r^{-k} A(a,r))' &= -k r^{-k-1} A(a,r) + r^{-k} A'(a,r) \\ &= -r^{-k-1}(r B'(a,r) + \underbrace{D(a,r)}_{=0,\text{ because } \delta V = 0}) + r^{-k}(B'(a,r) + C'(a,r)) \\ &= r^{-k} \underbrace{C'(a,r)}_{\geq 0}\end{aligned}$$

So $d(V, a, r) = \frac{1}{\alpha(k)} \frac{A(a,r)}{r^k}$ is not decreasing in r.
We also get the following identity for $0 < r_1 < r_2$:

$$\begin{aligned}\alpha(k)\big(d(V,a,r_2) - d(V,a,r_1)\big) &= \int_{r_1}^{r_2} r^{-k} A(a,r) \\ = \int_{r_1}^{r_2} r^{-k} C'(a,r)\, dr &= \int_{\{(x,S)\,:\, r_1 < |x-a| \leq r_2\}} |x-a|^{-k} \left| S^\perp\left(\frac{x-a}{|x-a|}\right)\right|^2 dV(x,S) \\ &= \int_{\{(x,S)\,:\, r_1 < |x-a| \leq r_2\}} |x-a|^{-k-2} |S^\perp(x-a)|^2 \, dV(x,S)\end{aligned}$$

$\square$

The monotonicity formula of course proves that $d(V, a)$ exists for every $a \in \mathbb{R}^n$ if V is stationary.

Next we will prove two results that will be used in the proof of isoperimetric inequality.

Theorem. If $0 < r_1 < r_2$, then $d(V,a,r_2) \exp\left[\int_{r_1}^{r_2} \frac{\|\delta V\|(B(a,r))}{\|V\|(B(a,r))} dr\right] \geq d(V,a,r_1)$ $\forall a \in \mathbb{R}^n$

Proof: Let $a \in \mathbb{R}^n$. We notice that $A'(a,r) = B'(a,r) + C'(a,r) \geq B'(a,r)$ for all $r \geq 0$. Combining this to the fact that $D(r) + rB'(r) = kA(r)$ for all $r \geq 0$ we see that

$$D(r) + rA'(r) \geq D(r) + rB'(r) = kA(r) \qquad \text{for all } r \geq 0$$

Thus $\dfrac{A'(a,r)}{A(a,r)} - \dfrac{k}{r} + \dfrac{D(a,r)}{rA(a,r)} \geq 0$ for all $r > 0$

It follows that

$$\frac{(d(V,a,r))'}{d(V,a,r)} = -\frac{\alpha(k)^{-1} k r^{-k-1} A(a,r)}{\alpha(k)^{-1} r^{-k} A(a,r)} + \frac{\alpha(k)^{-1} r^{-k} A'(a,r)}{\alpha(k)^{-1} r^{-k} A(a,r)}$$

$$= -\frac{k}{r} + \frac{A'(a,r)}{A(a,r)} \geq - \frac{D(a,r)}{rA(a,r)} \geq - \frac{\|\delta V\|(B(a,r))}{\|V\|(B(a,r))}$$

because $\dfrac{D(a,r)}{r} = \displaystyle\int_{B(a,r)} \langle \frac{x-a}{r}, \eta(x) \rangle \, d\|\delta V\| \leq \int_{B(a,r)} 1 \, d\|\delta V\| = \|\delta V\|(B(a,r))$

Integrating this inequality, the claim follows. $\square$

Recall the Besicovitch covering lemma:

Lemma. Let $\mathcal{B}$ be a family of closed balls in $\mathbb{R}^n$ whose union is bounded and let U be the set of centers of these balls. Then there exists such subfamilies $(\mathcal{B}_i)_{i=1}^{c(n)}$ of $\mathcal{B}$ that each subfamily is disjoint and $U \subset \bigcup_{i=1}^{c(n)} \bigcup_{B \in \mathcal{B}_i} B$. The number $c(n)$ of subfamilies depends only on dimension n.

Using this we can prove the following lemma.

Lemma. Let μ and ν be Radon measures on $\mathbb{R}^n$ with bounded supports. Assume that there exist such $U \subset \mathbb{R}^n$ and $\rho : U \to [0,\infty)$ that $\mu(\mathbb{R}^n \setminus U) = 0$ and

$$\mu(B(a,\rho(a))) \leq \nu(B(a,\rho(a))) \quad \text{for all } a \in U.$$

Then $\mu(\mathbb{R}^n) \leq c(n)\, \nu(\mathbb{R}^n)$.

Proof: Because μ has bounded support we may assume that U is bounded. Define $\mathcal{B} = \{ B(a,\rho(a)) : a \in U \}$. By Besicovitch covering lemma we have disjoint subfamilies $(\mathcal{B}_i)_{i=1}^{c(n)}$ which cover U. Then

$$\mu(\mathbb{R}^n) = \mu(U) \leq \mu\Big(\bigcup_{i=1}^{c(n)} \bigcup_{B \in \mathcal{B}_i} B\Big) \leq \sum_{i=1}^{c(n)} \sum_{B \in \mathcal{B}_i} \mu(B) \leq \sum_{i=1}^{c(n)} \sum_{B \in \mathcal{B}_i} \nu(B)$$

$$\overset{\text{Disjoint}}{=} \sum_{i=1}^{c(n)} \nu\Big(\bigcup_{B \in \mathcal{B}_i} B\Big) \leq \sum_{i=1}^{c(n)} \nu(\mathbb{R}^n) = c(n)\, \nu(\mathbb{R}^n) \qquad \square$$

Aischi Hrazak

Theorem (Isoperimetric inequality) Assume $\|V\|(\mathbb{R}^n) < \infty$ and $d(V,x) \geq 1$ for $\|V\|$-a.e. $x \in \mathbb{R}^n$. Then

$$\|V\|(\mathbb{R}^n)^{1-\frac{1}{k}} \leq C(k,n)\,\|\delta V\|(\mathbb{R}^n)$$

Proof Define $S = \left(\frac{2\|V\|(\mathbb{R}^n)}{\alpha(k)}\right)^{1/k}$ and $U = \{a \in \mathbb{R}^n : d(V,a) \geq 1\}$.
We note that

$$d(V,a,S) = \frac{\|V\|(B(a,S))}{\alpha(k) S^k} \leq \frac{\|V\|(\mathbb{R}^n)}{\alpha(k) S^k} = \frac{1}{2}$$

for every $a \in \mathbb{R}^n$. By the previous theorem we have

$$\exp \int_0^S \frac{\|\delta V\|(B(a,r))}{\|V\|(B(a,r))}\,dr \geq \frac{d(V,a)}{d(V,a,S)} \geq 2 \quad \text{for every } a \in U$$

So for every $a \in U$ there exists $\rho(a) \in [0,S)$ for which

$$S \frac{\|\delta V\|(B(a,\rho(a)))}{\|V\|(B(a,\rho(a)))} \geq \ln 2 \iff \|V\|(B(a,\rho(a))) \leq \frac{S}{\ln 2}\|\delta V\|(B(a,\rho(a)))$$

Using the previous lemma we see that

$$\|V\|(\mathbb{R}^n) \leq \frac{S}{\ln 2}\|\delta V\|(\mathbb{R}^n)$$

So $$\|V\|(\mathbb{R}^n)^{1-\frac{1}{k}} \leq \frac{2^{\frac{1}{k}}}{\alpha(k)^{\frac{1}{k}} \ln 2}\|\delta V\|(\mathbb{R}^n) =: C(n,k)\,\|\delta V\|(\mathbb{R}^n).$$

$\square$

If M^k is a C^2-submanifold of $\mathbb{R}^n$ then the isoperimetric inequality has the form

$$\|M\|(\mathbb{R}^n)^{1-\frac{1}{k}} \leq C(n,k)\left(\int |H|\,d\|M\| + \|\partial M\|(\mathbb{R}^n)\right)$$

where H is the mean curvature of M.