Geometric Measure Theory

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Contents

The material is collected mainly from books [EG], [Fe], [LY], [Ma], [Mo], and [Si] and from the lecture notes "Currents and varifolds" (fall 2011) by P. Mattila and "Moderni reaalianalyysi" by I. Holopainen.

The aim of the course is to give an introduction to the theory of varifolds and currents that are kind of generalized surfaces. They have been used in many geometric variational problems, in particular, in connections with higher dimensional minimal surfaces.

First we recall some basic notions of geometric measure theory.

1 Review of measure theory

1.1 Measures and outer measures

Let X be a set and let

$$\mathcal{P}(X) = \{A\colon A \subset X\}$$

be the power set of X (also denoted by 2^X).

Definition 1.2. The collection $\mathcal{M} \subset \mathcal{P}(X)$ is a σ-*algebra* "sigma algebra") in X if

(1) $\emptyset \in \mathcal{M}$;

(2) $A \in \mathcal{M} \;\Rightarrow\; A^c = X \setminus A \in \mathcal{M}$;

(3) $A_i \in \mathcal{M},\ i \in \mathbb{N} \;\Rightarrow\; \bigcup_{i=1}^{\infty} A_i \in \mathcal{M}$.

Example 1.3. 1. $\mathcal{P}(X)$ is the largest σ-algebra in X;

2. $\{\emptyset, X\}$ is the smallest σ-algebra in X;

3. $\mathrm{Leb}(\mathbb{R}^n)$ is the class of Lebesgue measurable sets of $\mathbb{R}^n$.

4. If $\mathcal{M}$ is a σ-algebra in X and $A \subset X$, then

$$\mathcal{M}|A = \{B \cap A\colon B \in \mathcal{M}\}$$

is a σ-algebra in A.

5. If $\mathcal{M}$ is a σ-algebra in X and $A \in \mathcal{M}$, then

$$\mathcal{M}_A = \{B \subset X : B \cap A \in \mathcal{M}\}$$

is a σ-algebra in X.

Definition 1.4. If $\mathcal{F} \subset \mathcal{P}(X)$ is a family of subsets of X, then

$$\sigma(\mathcal{F}) = \bigcap \{\mathcal{M}\colon \mathcal{M} \text{ is a } \sigma\text{-algebra in } X,\ \mathcal{F} \subset \mathcal{M}\}$$

is the σ-algebra *generated* by $\mathcal{F}$. It is the smallest σ-algebra that contains $\mathcal{F}$.

Example 1.5. Recall that the set $I \subset \mathbb{R}^n$ is an open n-interval if it is of the form

$$I = \{(x_1, \ldots, x_n)\colon a_j < x_j < b_j\},$$

where $-\infty \le a_j < b_j \le +\infty$. Then

$$\sigma(\{I\colon I\ n\text{-interval}\}) = \sigma(\{A\colon A \subset \mathbb{R}^n \text{ open}\}) \overset{\text{notat.}}{=} \mathrm{Bor}(\mathbb{R}^n)$$

is the σ-algebra of *Borel sets* of $\mathbb{R}^n$. (Can you prove the left side equality?)

Observe that all open subsets of $\mathbb{R}^n$, closed sets, $\mathcal{G}_\delta$ sets (countable intersections of open sets), $\mathcal{F}_\sigma$ sets (countable unions of closed sets), $\mathcal{F}_{\sigma\delta}$ sets, $\mathcal{G}_{\delta\sigma}$ sets (etc.) are Borel sets. Thus for example the set of rational numbers $\mathbb{Q}$ is Borel.

Remark 1.6. In every topological space X one can define Borel sets as

$$\mathrm{Bor}(X) = \sigma(\{A \colon A \subset X \text{ open}\}).$$

Definition 1.7. Let $\mathcal{M}$ be a σ-algebra in X. A mapping $\mu \colon \mathcal{M} \to [0, +\infty]$ is a *measure* if there holds:

(i) $\mu(\emptyset) = 0$,

(ii) $\mu\big(\bigcup_{i=1}^{\infty} A_i\big) = \sum_{i=1}^{\infty} \mu(A_i)$ if the sets $A_i \in \mathcal{M}$ are *disjoint*.

The triple $(X, \mathcal{M}, \mu)$ is called a *measure space* and the elements of $\mathcal{M}$ *measurable sets*.

The condition (ii) is called *countably additivity*. It follows from the definition that a measure is *monotone*: If $A, B \in \mathcal{M}$ and $A \subset B$, then $\mu(A) \le \mu(B)$.

Remark 1.8. 1. If $\mu(X) < \infty$, the measure μ is *finite*.

2. If $\mu(X) = 1$, then μ is *a probability measure*.

3. If $X = \bigcup_{i=1}^{\infty} A_i$, where $\mu(A_i) < \infty$ $\forall i$, the measure μ is σ*-finite*. Then we shall say that X is σ-finite with respect to μ.

4. If $A \in \mathcal{M}$ and $\mu(A) = 0$, then A is *of measure zero*.

5. If X is a topological space and $\mathrm{Bor}(X) \subset \mathcal{M}$ (i.e. every Borel set is measurable), then μ is a *Borel measure*.

Example 1.9. 1. $X = \mathbb{R}^n$, $\mathcal{M} = \mathrm{Leb}\,\mathbb{R}^n$ = the family of Lebesgue measurable sets and $\mu = m_n$ = the Lebesgue measure.

2. $X = \mathbb{R}^n$, $\mathcal{M} = \mathrm{Bor}\,\mathbb{R}^n$ = the family of Borel sets and $\mu = m_n | \mathrm{Bor}\,\mathbb{R}^n$ = the restriction of the Lebesgue measure to the family of Borel sets.

3. Let $X \neq \emptyset$ be any set. Fix $x \in X$ and set for all $A \subset X$

$$\mu(A) = \begin{cases} 1, & \text{if } x \in A; \\ 0, & \text{if } x \notin A. \end{cases}$$

Then $\mu \colon \mathcal{P}(X) \to [0, +\infty]$ is a measure (so called *Dirac measure* at $x \in X$). We often write $\mu = \delta_x$.

4. If $f \colon \mathbb{R}^n \to [0, +\infty]$ is Lebesgue measurable, then $\mu \colon \mathrm{Leb}(\mathbb{R}^n) \to [0, +\infty]$,

$$\mu(E) = \int_E f(x)\, dm_n(x),$$

is a measure.

5. If $(X, \mathcal{M}, \mu)$ is a measure space and $A \in \mathcal{M}$, then the mapping $\mu\llcorner A\colon \mathcal{M}_A \to [0, +\infty]$,
$$(\mu\llcorner A)(B) = \mu(B \cap A),$$
is a measure. It is called *the restriction of μ to A.*

Theorem 1.10. *Let $(X, \mathcal{M}, \mu)$ be a measure space and $A_1, A_2, \ldots \in \mathcal{M}$.*

(a) If $A_1 \subset A_2 \subset A_3 \cdots$, then
$$\mu\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \lim_{i\to\infty} \mu(A_i).$$

(b) If $A_1 \supset A_2 \supset A_3 \supset \cdots$ and $\mu(A_k) < \infty$ for some k, then
$$\mu\Big(\bigcap_{i=1}^{\infty} A_i\Big) = \lim_{i\to\infty} \mu(A_i).$$

Proof. Course "Mitta ja integraali". □

Definition 1.11. A mapping $\tilde{\mu}\colon \mathcal{P}(X) \to [0, +\infty]$ is *an outer measure* in X if the following holds:

(i) $\tilde{\mu}(\emptyset) = 0$;

(ii) $\tilde{\mu}(A) \le \sum_{i=1}^{\infty} \tilde{\mu}(A_i)$ if $A \subset \bigcup_{i=1}^{\infty} A_i \subset X$.

Remark 1.12. 1. An outer measure is defined for <u>all</u> subsets of X.

2. Condition (ii) (monotone subadditivity) implies that an outer measure is monotone, i.e. $\tilde{\mu}(A) \le \tilde{\mu}(B)$ if $A \subset B \subset X$.

3. In many books an outer measure is simply called a measure. (Soon we will do so, too.)

4. Let $\tilde{\mu}$ be an outer measure in X and $A \subset X$. Then the *restriction of $\tilde{\mu}$ to A*, defined by
$$(\tilde{\mu}\llcorner A)(B) = \tilde{\mu}(B \cap A)$$
is an outer measure in X.

Every outer measure defines the σ-algebra of "measurable" sets in terms of the Carathéodory condition.

Definition 1.13. Let $\tilde{\mu}$ be an outer measure in X. A set $E \subset X$ is *$\tilde{\mu}$-measurable*, or briefly *measurable*, if
$$\tilde{\mu}(A) = \tilde{\mu}(A \cap E) + \tilde{\mu}(A \setminus E)$$
for all $A \subset X$.

Theorem 1.14. *Let $\tilde{\mu}$ be an outer measure in X and*
$$\mathcal{M} = \mathcal{M}_{\tilde{\mu}} = \{E \subset X : E \text{ is } \tilde{\mu}\text{-measurable}\}$$
Then

(a) $\mathcal{M}$ is a σ-algebra and

(b) $\mu = \tilde{\mu}|\mathcal{M}$ is a measure (i.e. μ is countably additive).

Proof. Course "Mitta ja integraali". □

Definition 1.15. We say that an outer measure $\tilde{\mu}$ in a topological space X is *a Borel outer measure* if every Borel set of X is $\tilde{\mu}$-measurable (i.e. if the measure defined by $\tilde{\mu}$ is a Borel measure).

1.16 Metric outer measure

We shall next study the question when an outer measure $\tilde{\mu}$ in a topological space X is Borel.

Definition 1.17 (Carathéodory's criterion)**.** An outer measure $\tilde{\mu}$ in a metric space (X, d) is *a metric outer measure* if

$$\tilde{\mu}(A \cup B) = \tilde{\mu}(A) + \tilde{\mu}(B)$$

for all $A, B \subset X$, for which $\operatorname{dist}(A, B) = \inf\{d(a, b) \colon a \in A, b \in B\} > 0$.

Theorem 1.18. *An outer measure $\tilde{\mu}$ of a metric space (X, d) is a Borel outer measure if and only if $\tilde{\mu}$ is a metric outer measure.*

We first formulate and prove the following lemma.

Lemma 1.19. *Let $\tilde{\mu}$ be a metric outer measure, $A \subset X$ and G an open set such that $A \subset G$. If*

$$A_k = \{x \in A \colon \operatorname{dist}(x, G^c) \geq 1/k\}, \ k \in \mathbb{N},$$

then $\tilde{\mu}(A) = \lim_{k\to\infty} \tilde{\mu}(A_k)$.

Proof. Since G is open, $A \subset \bigcup_{k=1}^{\infty} A_k$. Thus $A = \bigcup_{k=1}^{\infty} A_k$. Let

$$B_k = A_{k+1} \setminus A_k.$$

Then

$$A = A_{2n} \cup \left(\bigcup_{k=n}^{\infty} B_{2k} \right) \cup \left(\bigcup_{k=n}^{\infty} B_{2k+1} \right),$$

and thus

$$\tilde{\mu}(A) \leq \tilde{\mu}(A_{2n}) + \underbrace{\sum_{k=n}^{\infty} \tilde{\mu}(B_{2k})}_{=(I)} + \underbrace{\sum_{k=n}^{\infty} \tilde{\mu}(B_{2k+1})}_{=(II)}.$$

Let now $n \to \infty$.

(1) If the sums $(I), (II) \to 0$ as $n \to \infty$, then

$$\tilde{\mu}(A) \leq \lim_{n\to\infty} \tilde{\mu}(A_{2n}) \leq \tilde{\mu}(A)$$

and the claim is true.

(2) If $(I) \not\to 0$ as $n \to \infty$, then

$$\sum_k \tilde{\mu}(B_{2k}) = \infty.$$

On the other hand,

$$A \supset A_{2n} \supset \bigcup_{k=1}^{n-1} B_{2k},$$

where

$$\operatorname{dist}(B_{2k}, B_{2k+2}) \geq \frac{1}{2k+1} - \frac{1}{2k+2} > 0.$$

Because $\tilde{\mu}$ is a metric outer measure, we have

$$\sum_{k=1}^{n-1} \tilde{\mu}(B_{2k}) = \tilde{\mu}\big(\bigcup_{k=1}^{n-1} B_{2k}\big) \leq \tilde{\mu}(A_{2n}) \leq \tilde{\mu}(A).$$

Letting $n \to \infty$ we obtain

$$\tilde{\mu}(A) = \lim_{k\to\infty} \tilde{\mu}(A_k) = \infty.$$

The argument goes in the same way if the sum $(II) \not\to 0$ as $n \to \infty$. $\square$

Proof of Theorem 1.18. Suppose first that $\tilde{\mu}$ is a metric outer measure. We want to prove that $\tilde{\mu}$ is a Borel outer measure. Because $\mathrm{Bor}(X) = \sigma(\{F \colon F \subset X \text{ closed}\})$ and $\mathcal{M}_{\tilde{\mu}}$ is a σ-algebra, it is enough to show that every closed set $F \subset X$ is $\tilde{\mu}$-measurable.

Let $E \subset X$ be an arbitrary test set in the Carathéodory condition. We apply Lemma 1.19 for the sets $A = E \setminus F$ and $G = X \setminus F$. Let $A_k = \{x \in E \setminus F \colon \operatorname{dist}(x, G^c) \geq 1/k\},\ k \in \mathbb{N}$. Then

$$\operatorname{dist}(A_k, F) \geq 1/k$$

and

$$\lim_{k\to\infty} \tilde{\mu}(A_k) = \tilde{\mu}(E \setminus F).$$

Because $\tilde{\mu}$ is metric,

$$\tilde{\mu}(E) \geq \tilde{\mu}\big((E \cap F) \cup A_k\big) = \tilde{\mu}(E \cap F) + \tilde{\mu}(A_k).$$

Letting $k \to \infty$ we get

$$\tilde{\mu}(E) \geq \tilde{\mu}(E \cap F) + \tilde{\mu}(E \setminus F).$$

On the other hand it follows from the monotonicity of the outer measure that

$$\tilde{\mu}(E) \leq \tilde{\mu}(E \cap F) + \tilde{\mu}(E \setminus F).$$

Thus F is $\tilde{\mu}$-measurable and $\tilde{\mu}$ is a Borel outer measure.

The proof of the converse implication is left as an exercise. $\square$

1.20 Regularity of measures, Radon-measures

Among outer measures particularly useful are those with a large class of measurable sets. Such outer measures are called regular.

Definition 1.21. We say that an outer measure $\tilde{\mu}$ of X is *regular* if, for every $A \subset X$, there exists a $\tilde{\mu}$-measurable set E such that $A \subset E$ and $\mu(E) = \tilde{\mu}(A)$ (such a set E is called *a measurable cover* of A.)

Definition 1.22. Let X be a topological space.

(a) We say that an outer measure $\tilde{\mu}$ of X is *Borel regular* if μ is a Borel measure and for every $A \subset X$ there exists a Borel set $B \in \mathrm{Bor}(X)$ such that $A \subset B$ and $\mu(B) = \tilde{\mu}(A)$.

(b) Let $(X, \mathcal{M}, \mu)$ be a measure space such that $\mathrm{Bor}(X) \subset \mathcal{M}$ (i.e. μ is a Borel measure). Then the measure μ is called *Borel regular* if for every $A \in \mathcal{M}$ there exists $B \in \mathrm{Bor}(X)$ such that $A \subset B$ and $\mu(A) = \mu(B)$.

Lemma 1.23. *If $\tilde{\mu}$ is a Borel regular outer measure in X and $A \subset X$ is $\tilde{\mu}$-measurable s.t. $\mu(A) < \infty$, then $\tilde{\mu} \llcorner A$ is Borel regular. If $A \in \mathrm{Bor}(X)$, then the assumption $\mu(A) < \infty$ is not needed.*

Proof. Exercise. $\square$

Theorem 1.24. *Let $\tilde{\mu}$ be a Borel regular outer measure in a metric space X, $A \subset X$ $\tilde{\mu}$-measurable and $\varepsilon > 0$.*

(a) If $\mu(A) < \infty$, then there exists a closed set $C \subset A$ s.t. $\mu(A \setminus C) < \varepsilon$.

(b) If there exist open sets $V_1, V_2, \ldots \subset X$ s.t. $A \subset \bigcup_{i=1}^{\infty} V_i$ and $\mu(V_i) < \infty$ $\forall i$, then there exists an open set $V \subset X$ s.t. $A \subset V$ and $\mu(V \setminus A) < \varepsilon$.

Proof. (a): By replacing $\tilde{\mu}$ with a Borel regular outer measure $\tilde{\mu} \llcorner A$ (see Lemma 1.23) we may assume that $\tilde{\mu}(X) < \infty$. We first prove the claim for Borel sets A. Let

$$\mathcal{D} = \{A \subset X \colon \forall \varepsilon > 0 \; \exists \text{ closed } C \subset A \text{ and open } V \supset A \text{ s.t. } \mu(V \setminus C) < \varepsilon\}.$$

We easily see that $\mathcal{D}$ satisfies condition (1) and (2) in the definition of a σ-algebra. Suppose that $A_1, A_2, \ldots \in \mathcal{D}$ and let $\varepsilon > 0$. Then there exists closed sets C_i and open sets V_i s.t. $C_i \subset A_i \subset V_i$ and $\mu(V_i \setminus C_i) < \varepsilon/2^i$. Now $V = \bigcup_i V_i$ is open and

$$\mu(\underbrace{V \setminus \bigcup_i C_i}_{\subset \bigcup_i (V_i \setminus C_i)}) \le \sum_i \mu(V_i \setminus C_i) < \varepsilon.$$

On the other hand, by Theorem 1.10 (b)

$$\lim_{n \to \infty} \mu(V \setminus \bigcup_{i=1}^{n} C_i) = \mu(V \setminus \bigcup_{i=1}^{\infty} C_i),$$

and hence there exists n s.t.

$$\mu(V \setminus \bigcup_{i=1}^{n} C_i) < \varepsilon.$$

Because $\bigcup_{i=1}^{n} C_i$ is closed, $\mathcal{D}$ satisfies also the condition (3) of a σ-algebra. We next show that $\mathcal{D}$ contains closed sets. Let C be closed and

$$V_i = \{x \in X \colon \operatorname{dist}(x, C) < 1/i\}.$$

Then V_i is open, $V_1 \supset V_2 \supset \cdots$, and $C = \bigcap_i V_i$. Therefore $\lim_{i\to\infty} \mu(V_i) = \mu(C)$ and $\lim_{i\to\infty} \mu(V_i \setminus C) = 0$. This implies that $C \in \mathcal{D}$. Thus $\mathcal{D}$ is a σ-algebra containing all closed sets. In particular, $\operatorname{Bor}(X) \subset \mathcal{D}$. Therefore part (a) holds for all Borel sets.

Suppose next that A is $\tilde{\mu}$-measurable and $\mu(A) < \infty$. Because $\tilde{\mu}$ is Borel regular, there exists a Borel set $B \supset A$ s.t. $\mu(A) = \mu(B)$. Then $\mu(B \setminus A) = 0$. Furthermore, there exists a Borel set $D \supset B \setminus A$ s.t. $\mu(D) = 0$. Now $E = B \setminus D$ is Borel, $E \subset A$, and

$$\mu(\underbrace{A \setminus E}_{\subset D}) = 0.$$

Applying the first part of the proof to the Borel set E we conclude that, for each $\varepsilon > 0$, there exists a closed set $C \subset E = B \setminus D$ $(\subset A)$ s.t. $\mu(E \setminus C) < \varepsilon$. But then

$$\mu(A \setminus C) \le \mu(A \setminus E) + \mu(E \setminus C) < \varepsilon,$$

and hence (a) holds for the set A.

(b) By applying part (a) to the set $V_i \setminus A$ we obtain closed sets $C_i \subset V_i \setminus A$ s.t.

$$\mu(V_i \setminus A \setminus C_i) < \varepsilon 2^{-i}.$$

Then $V = \bigcup_i (V_i \setminus C_i)$ is open, $A \subset V$ and $\mu(V \setminus A) < \varepsilon$. □

Remark 1.25. The Borel regularity of the outer measure *was not* needed to prove claims (a) and (b) for Borel sets A. Therefore Theorem 1.24 holds for all Borel outer measures, if we, furthermore, assume that A is Borel.

We also have the following version of Theorem 1.24

Theorem 1.26. *Let $\tilde{\mu}$ be a Borel regular outer measure in a metric space X and*

$$X = \bigcup_{j=1}^{\infty} V_j,$$

where V_j is open and $\mu(V_j) < \infty$ for each $j \in \mathbb{N}$. Then

$$\tilde{\mu}(A) = \inf\{\mu(U) \colon U \text{ open}, A \subset U\} \tag{1.27}$$

for every $A \subset X$, and

$$\mu(A) = \sup\{\mu(C) \colon C \text{ closed}, C \subset A\} \tag{1.28}$$

for every μ-measurable $A \subset X$.

So called *Radon measures* will be important in what follows. These will be defined next. Recall that a topological space X is *locally compact*, if every point $x \in X$ has a neighbourhood with compact closure. A topological space is *Hausdorff*, if its distinct points have disjoint neighbourhoods.

Definition 1.29. Let X be a locally compact Hausdorff space. We say that a measure μ is a *Radon measure*, if μ is a Borel measure and

(a) $\mu(K) < \infty$ for all compact $K \subset X$;

(b) $\mu(V) = \sup\{\mu(K) \colon K \subset V \text{ compact}\}$ for all open $V \subset X$;

(c) $\mu(B) = \inf\{\mu(V) \colon B \subset V \text{ and } V \subset X \text{ open}\}$ for all Borel sets $B \in \mathrm{Bor}(X)$.

Remark 1.30.

1. In general, a Borel regular measure (in a locally compact Hausdorff space) need not be a Radon measure.

2. On the other hand a Radon measure need not be Borel regular: Let $A \subset \mathbb{R}$ be non-Lebesgue measurable, $\tilde{\mu} = m^* \llcorner A$ and

$$\mu = \tilde{\mu}|\{E \subset \mathbb{R} \colon E \ \tilde{\mu}\text{-measurable}\}.$$

Then μ is a Radon measure, but not Borel regular.

In some cases Radon measures can be easily characterised.

Theorem 1.31. *Let μ be a Borel measure in $\mathbb{R}^n$. Then μ is a Radon measure, if and only if μ is* locally finite, *i.e.*

$$\forall x \in \mathbb{R}^n, \ \mu\big(B(x,r)\big) < \infty, \ \text{when } 0 < r < r_x.$$

Proof. It follows from Definition 1.29 (a) that all Radon measures are locally finite.

Suppose next that μ is locally finite Borel measure in $\mathbb{R}^n$. If $K \subset \mathbb{R}^n$ is compact, then choose for every $x \in K$ an open ball with centre at x with a finite measure. Because K is compact, it can be covered with finitely many such balls. Therefore the measure of K is finite and (a) holds.

We next prove conditions (b) and (c) for every Borel set $A \subset \mathbb{R}^n$. By applying part (a) of Theorem 1.24 (see also Remark 1.25) for Borel sets A_i of finite measure,

$$A_i = A \cap B(0,i), \quad B(0,i) = \{x \in \mathbb{R}^n \colon |x| \le i\},$$

we find closed sets $C_i \subset A_i$ s.t.

$$\mu(A_i \setminus C_i) < 1/i.$$

Then C_i is a compact set as a closed and bounded set (in $\mathbb{R}^n$). Now

$$\mu(A) \ge \mu(A_i) \ge \mu(C_i) > \mu(A_i) - 1/i \xrightarrow{1.10(a)} \mu(A)\,.$$

This implies (b). Because $A \subset \bigcup_i B(0,i)$ and $\mu(B(0,i)) < \infty$, it follows from Theorem 1.24 part (b) that there exist open sets $V_j \subset \mathbb{R}^n$ s.t. $A \subset V_j$ and $\mu(V_j \setminus A) < 1/j$. Then

$$\mu(A) \le \mu(V_j) = \mu(A) + \mu(V_j \setminus A) < \mu(A) + 1/j,$$

This implies (c). □

Corollary 1.32. *Let $\tilde{\mu}$ be a locally finite metric outer measure in $\mathbb{R}^n$. Then the measure $\mu = \tilde{\mu}|\mathcal{M}$, $\mathcal{M} = \{A \subset \mathbb{R}^n \colon A\ \tilde{\mu}\text{-measurable}\}$, determined by $\tilde{\mu}$ is a Radon measure.*

Remark 1.33. Theorem 1.31 holds also more generally. For example, if X is a locally compact metric space, whose topology has a countable base.

Convention: From now on we call an outer measure $\tilde{\mu}$ simply a measure and (to simplify the notation) we denote it by μ.

Note that outer measures and measures come in a sense hand in hand. Indeed, an outer measure $\tilde{\mu} \colon \mathcal{P}(X) \to [0,+\infty]$ defines the measure $\mu = \tilde{\mu}|\mathcal{M}$, where $\mathcal{M}$ is the σ-algebra of $\tilde{\mu}$-measurable sets and, on the other hand, every measure $\mu \colon \mathcal{M} \to [0,+\infty]$ defined on a σ-algebra $\mathcal{M} \subset \mathcal{P}(X)$ can be extended to an outer measure $\tilde{\mu} \colon \mathcal{P}(X) \to [0,+\infty]$ by setting

$$\tilde{\mu}(A) = \inf\{\mu(B) \colon A \subset B \in \mathcal{M}\}.$$

Let μ be regular and $A_i \subset A_{i+1} \subset X$ for $i \in \mathbb{N}$. We have the counterpart of Theorem 1.10 (a)

$$\mu\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \lim_{i \to \infty} \mu(A_i)$$

even if the sets A_i are not assumed to be μ-measurable.

Let then X be a locally compact, separable metric space. We say that μ is a *Radon (outer) measure* if μ is Borel regular and if μ is finite on compact subsets of X. Then such a measure μ has the properties

$$\mu(A) = \inf\{\mu(U) \colon U \text{ open}, A \subset U\}$$

for every $A \subset X$ and

$$\mu(A) = \sup\{\mu(K) \colon K \text{ compact}, K \subset A\}$$

for every μ-measurable $A \subset X$.

Since μ is finite on compact sets, we can integrate continuous functions with compact support. In particular, if H is a Hilbert space with inner product $(\cdot,\cdot)$ and if $C_0(X,H)$ denotes the space of continuous functions $X \to H$ with compact support, then associated to each Radon measure μ and each μ-measurable H-valued function $\nu\colon X \to H$, with $|\nu| = 1$ μ-a.e., we have the linear functional $L\colon C_0(X,H) \to \mathbb{R}$ defined by

$$L(f) = \int_X (f,\nu)\, d\mu.$$

The following *Riesz representation theorem* gives the converse:

Theorem 1.34. *Let H be a finite dimensional Hilbert space and let $L\colon C_0(X,H) \to \mathbb{R}$ be a linear functional such that*

$$\sup\{L(f)\colon f \in C_0(X,H), |f| \le 1, \operatorname{supp} f \subset K\} < \infty$$

for each compact $K \subset X$. Then there exist a Radon measure μ and a μ-measurable mapping $\nu\colon X \to H$ such that $|\nu(x)| = 1$ for μ-a.e. $x \in X$ and

$$L(f) = \int_X (f,\nu)\, d\mu$$

for every $f \in C_0(X,H)$.

We will return to this later.

1.35 Hausdorff measure

The Lebesgue n-dimensional measure m_n is well suited for the measurement of the size of “large” subsets of $\mathbb{R}^n$, but it is too crude for measuring “small” subsets of $\mathbb{R}^n$. For example, m_2 cannot distinguish a singleton of $\mathbb{R}^2$ from a line, because both have measure zero.

In this chapter we introduce a whole spectrum of “s-dimensional” measures $\mathcal{H}^s$, $0 \le s < \infty$, which are able to see the fine structure of sets, better than the Lebesgue measure. The key idea is that a set $A \subset \mathbb{R}^n$ is “s-dimensional”, if $0 < \mathcal{H}^s(A) < \infty$, even if the geometric structure of A were very complicated.

These measures can be defined in any metric space (X,d). We suppose, however, that X is *separable*, i.e. X has a countable dense subset $S = \{x_i\}_{i=1}^\infty$, and hence $X = \bar{S}$. This assumption is only needed to guarantee that X has so called δ-covering for all $\delta > 0$.

Definition 1.36. 1. The *diameter* of a nonempty set $E \subset X$ is

$$d(E) = \sup_{x,y\in E} d(x,y).$$

2. A countable collection $\{E_i\}_{i=1}^\infty$ of subsets of X is a *δ-covering*, $\delta > 0$, of $A \subset X$ if

$$A \subset \bigcup_{i=1}^\infty E_i \quad \text{and } d(E_i) \le \delta \ \forall i \in \mathbb{N}.$$

We fix a “dimension” $s \in [0,\infty)$ and $\delta > 0$. For $A \subset X$, we define

$$\mathcal{H}^s_\delta(A) = \inf\Big\{\omega_s \sum_{i=0}^\infty \big(d(E_i)/2\big)^s \colon \{E_i\} \text{ is a } \delta\text{-covering of } A\Big\}, \tag{1.37}$$

where ω_s is the volume of the unit ball in $\mathbb{R}^s$ in case s is a positive integer and otherwise some convenient positive constant, and where we make the convention that $\big(d(\{x\})/2\big)^0 = 1\ \forall x \in X$ and $\big(d(\emptyset)/2\big)^s = 0\ \forall s \geq 0$.

We readily see from the definition that

$$\mathcal{H}^s_{\delta_1}(A) \geq \mathcal{H}^s_{\delta_2}(A),$$

if $0 < \delta_1 \leq \delta_2$. Therefore the following limit (1.39) exists and we can set the definition.

Definition 1.38. The *s-dimensional Hausdorff (outer) measure* of a set $A \subset X$ is

$$\mathcal{H}^s(A) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(A) \quad \Big(= \sup_{\delta > 0} \mathcal{H}^s_\delta(A)\Big). \tag{1.39}$$

Remark 1.40. The constant ω_s above is usually chosen as

$$\omega_s = \frac{\pi^{s/2}}{\Gamma\big(\frac{s}{2}+1\big)},$$

where $\Gamma(t) = \int_0^\infty e^{-x}x^{t-1}\,dx,\ 0 < t < \infty$, is the usual gamma function.

In particular, this guarantees that $\mathcal{H}^n$ and the n-dimensional Lebesgue outer measure m_n^* coincide in $\mathbb{R}^n$, i.e.

$$\mathcal{H}^n(A) = m_n^*(A) \quad \forall A \subset \mathbb{R}^n.$$

We will not prove this identity. For the proof, see e.g. [Si].

Theorem 1.41. *(i) $\mathcal{H}^s_\delta \colon \mathcal{P}(X) \to [0, +\infty]$ is an outer measure for all $\delta > 0$.*

(ii) $\mathcal{H}^s \colon \mathcal{P}(X) \to [0, +\infty]$ is a metric outer measure.

Proof. (i) (a) Clearly $\mathcal{H}^s_\delta(\emptyset) = 0$.

(b) Let then $A \subset \bigcup_{i=1}^\infty A_i \subset X$. We may suppose that $\mathcal{H}^s_\delta(A_i) < \infty\ \forall i$. Let $\varepsilon > 0$ and choose for every i a δ-covering $\{E^i_j\}_{j=1}^\infty$ of the set A_i s.t.

$$\omega_s \sum_{j=1}^\infty \big(d(E^i_j)/2\big)^s \leq \mathcal{H}^s_\delta(A_i) + \varepsilon 2^{-i}.$$

Then $\bigcup_{i,j} E^i_j$ is a δ-covering of the union $\bigcup_{i=1}^\infty A_i$ and thus also of A and therefore

$$\begin{aligned}\mathcal{H}^s_\delta(A) &\leq \omega_s \sum_{i,j} \big(d(E^i_j)/2\big)^s \\ &\leq \sum_{i=1}^\infty \big(\mathcal{H}^s_\delta(A_i) + \varepsilon 2^{-i}\big) \\ &\leq \varepsilon + \sum_{i=1}^\infty \mathcal{H}^s_\delta(A_i).\end{aligned}$$

Letting $\varepsilon \to 0$ the desired conclusion follows.

(ii) Clearly $\mathcal{H}^s(\emptyset) = 0$. If $A \subset \bigcup_{i=1}^\infty A_i \subset X$, then by part (i) and the definition of $\mathcal{H}^s$ we obtain

$$\mathcal{H}^s_\delta(A) \leq \sum_{i=1}^\infty \mathcal{H}^s_\delta(A_i) \leq \sum_{i=1}^\infty \mathcal{H}^s(A_i).$$

Letting $\delta \to 0$ we see that $\mathcal{H}^s$ is an outer measure. Let then $A_1, A_2 \subset X$ be sets, for which $\operatorname{dist}(A_1, A_2) > 0$. We wish to show that

$$\mathcal{H}^s(A_1 \cup A_2) = \mathcal{H}^s(A_1) + \mathcal{H}^s(A_2).$$

It is enough to show that

$$\mathcal{H}^s_\delta(A_1 \cup A_2) \geq \mathcal{H}^s_\delta(A_1) + \mathcal{H}^s_\delta(A_2), \tag{1.42}$$

if $\delta \leq \operatorname{dist}(A_1, A_2)/3$. We may assume that $\mathcal{H}^s_\delta(A_1 \cup A_2) < \infty$. Let $\varepsilon > 0$ and choose a δ-covering $\{E_i\}_{i=1}^\infty$ of the set $A_1 \cup A_2$ such that

$$\omega_s \sum_{i=1}^\infty \big(d(E_i)/2\big)^s \leq \mathcal{H}^s_\delta(A_1 \cup A_2) + \varepsilon.$$

Because $\delta \leq \operatorname{dist}(A_1, A_2)/3$, every E_i intersects at most one of the sets A_1 or A_2. Therefore we may divide the δ-covering $\{E_i\}_{i=1}^\infty$ of $A_1 \cup A_2$ into two *disjoint* δ-coverings of A_1 and A_2 as

$$\{E_i\}_{i=1}^\infty = \{E_i'\}_{i=1}^\infty \sqcup \{E_i''\}_{i=1}^\infty,$$

where

$$A_1 \subset \bigcup_{i=1}^\infty E_i' \quad \text{and} \quad A_2 \subset \bigcup_{i=1}^\infty E_i''.$$

Therefore

$$\begin{aligned} \mathcal{H}^s_\delta(A_1) + \mathcal{H}^s_\delta(A_2) &\leq \omega_s \sum_{i=1}^\infty \big(d(E_i')/2\big)^s + \omega_s \sum_{i=1}^\infty \big(d(E_i'')/2\big)^s \\ &= \omega_s \sum_{i=1}^\infty \big(d(E_i)/2\big)^s \\ &\leq \mathcal{H}^s_\delta(A_1 \cup A_2) + \varepsilon. \end{aligned}$$

Because $\varepsilon > 0$ was arbitrary, we obtain (1.42).

□

On the basis of Theorems 1.18 and 1.41 every Borel-set of X is $\mathcal{H}^s$-measurable. We denote the restriction of $\mathcal{H}^s$ to $\mathcal{H}^s$-measurable sets with the same symbol $\mathcal{H}^s$. Now there holds:

Theorem 1.43. *$\mathcal{H}^s$ is a Borel-measure.*

Corollary 1.32 yields now:

Corollary 1.44. *If $A \subset \mathbb{R}^n$ is $\mathcal{H}^s$-measurable and $\mathcal{H}^s(A) < \infty$, then $\mathcal{H}^s \llcorner A$ is a Radon-measure.*

Theorem 1.45. *The outer measure $\mathcal{H}^s$ of a separable metric space X is Borel-regular.*

Proof. Because by the previous theorem the outer measure defined by $\mathcal{H}^s$ is Borel, it is enough to show that for all $A \subset X$ there exists $B \in \mathrm{Bor}(X)$ s.t. $A \subset B$ and $\mathcal{H}^s(A) = \mathcal{H}^s(B)$.

Let $A \subset X$. If $\mathcal{H}^s(A) = \infty$, we may choose $B = X$ and the claim holds. If $\mathcal{H}^s(A) < \infty$, then we choose a $1/i$-covering $\{E^i_j\}_{j=1}^\infty$ of A for each $i \in \mathbb{N}$ s.t.

$$\omega_s \sum_{j=1}^\infty \big(d(E^i_j)/2\big)^s \le \mathcal{H}^s_{1/i}(A) + 1/i.$$

Because $d(E) = d(\bar{E})$ for all $E \subset X$, we may suppose that the sets E^i_j are closed. Then

$$B = \bigcap_{i=1}^\infty \bigcup_{j=1}^\infty E^i_j$$

is a Borel set and $A \subset B$. Furthermore, $\{E^i_j\}$ is a $1/i$-covering of B for all $i \in \mathbb{N}$, and hence

$$\mathcal{H}^s_{1/i}(A) \le \mathcal{H}^s_{1/i}(B) \le \omega_s \sum_{j=1}^\infty \big(d(E^i_j)/2\big)^s \le \mathcal{H}^s_{1/i}(A) + 1/i.$$

Letting $i \to \infty$ the claim $\mathcal{H}^s(A) = \mathcal{H}^s(B)$ follows. □

Remark 1.46.

1. $\mathcal{H}^0$ is the counting measure.
2. $\mathcal{H}^s_\delta$ is not, in general, a metric outer measure.
3. *Roughly* speaking $\mathcal{H}^1 \sim$ is a length measure, $\mathcal{H}^2 \sim$ is area, etc.
4. It is easily seen that (e.g.) the plane $\mathbb{R}^2$ is not σ-finite with respect to $\mathcal{H}^1$.

1.47 Hausdorff dimension

Let (X, d) be a separable metric space. In this chapter we shall define a dimension for sets $A \subset X$, which reflects the metric size of the set A. Unlike the topological dimension, this dimension need not be an integer.

Lemma 1.48. *Let $A \subset X$ and $s \ge 0$.*

(i) If $\mathcal{H}^s(A) < \infty$, then $\mathcal{H}^t(A) = 0$ for all $t > s$.

(ii) If $\mathcal{H}^s(A) > 0$, then $\mathcal{H}^t(A) = \infty$ for all $0 \le t < s$.

Proof. It is enough to prove (i), because the claim (ii) follows from (i). Let $\delta > 0$ and $\{E_j\}_{j=1}^\infty$ be a δ-covering of A s.t.

$$\omega_s \sum_{j=1}^\infty \big(d(E_j)/2\big)^s \le \mathcal{H}^s_\delta(A) + 1 \le \mathcal{H}^s(A) + 1 < \infty.$$

Then for all $t > s$

$$\begin{aligned}\mathcal{H}^t_\delta(A) &\le \omega_t \sum_{j=1}^{\infty} \big(d(E_j)/2\big)^t \\ &= \omega_t \sum_{j=1}^{\infty} \big(d(E_j)/2\big)^s \big(d(E_j)/2\big)^{t-s} \\ &\le \frac{\omega_t}{\omega_s} (\delta/2)^{t-s} \omega_s \sum_{j=1}^{\infty} \big(d(E_j)/2\big)^s \\ &\le \frac{\omega_t}{\omega_s} (\delta/2)^{t-s} \left(\mathcal{H}^s(A) + 1\right).\end{aligned}$$

The claim follows by letting $\delta \to 0$. □

Definition 1.49. The *Hausdorff dimension* of a subset $A \subset X$ is a number

$$\dim_{\mathcal{H}}(A) = \inf\{s > 0 \colon \mathcal{H}^s(A) = 0\}.$$

Summarizing what was said above:

1. If $t < \dim_{\mathcal{H}}(A)$, then $\mathcal{H}^t(A) = \infty$.
2. If $t > \dim_{\mathcal{H}}(A)$, then $\mathcal{H}^t(A) = 0$.

In general, about the value $\mathcal{H}^s(A)$, for $s = \dim_{\mathcal{H}}(A)$, we cannot say anything: it can take any value in $[0, \infty]$. Nevertheless:

$$0 < \mathcal{H}^s(A) < \infty \;\Rightarrow\; \dim_{\mathcal{H}}(A) = s\,. \tag{1.50}$$

A set $A \subset X$, for which $0 < \mathcal{H}^s(A) < \infty$ holds, is called an *s-set*.

Lemma 1.51. *(i) If $A \subset B$, then $\dim_{\mathcal{H}}(A) \le \dim_{\mathcal{H}}(B)$.*

(ii) If $A_k \subset X,\ k \in \mathbb{N}$, then

$$\dim_{\mathcal{H}}\Big(\bigcup_{k=1}^{\infty} A_k\Big) = \sup_k \dim_{\mathcal{H}}(A_k).$$

Proof. (Exerc.) □

Thus, for example, $\dim_{\mathcal{H}}(\mathbb{Q}) = 0$.

1.52 Hausdorff measures in $\mathbb{R}^n$

Next we evaluate (or rather estimate) Hausdorff measures and dimensions of Cantor type fractal sets in $\mathbb{R}^n$. To this end we study the invariance properties of Hausdorff measures. There are other, more efficient, methods for the determination of the Hausdorff dimension but these will not be discussed in this course.

Recall first that a mapping $T \colon \mathbb{R}^n \to \mathbb{R}^n$ is an *isometry*, if

$$|Tx - Ty| = |x - y| \quad \forall x, y \in \mathbb{R}^n.$$

It is well-known that every isometry of $\mathbb{R}^n$ is an *affine* mapping, i.e. of the form

$$Tx = a_0 + Ux,$$

where $a_0 \in \mathbb{R}^n$ and $U\colon \mathbb{R}^n \to \mathbb{R}^n$ is a linear isometry.

In the same way, a mapping $R\colon \mathbb{R}^n \to \mathbb{R}^n$ is said to be a *similarity*, if

$$|Rx - Ry| = c|x - y| \quad \forall x, y \in \mathbb{R}^n,$$

where $c > 0$ is a constant (stretching factor, scaling factor, etc.). Then R is of the form

$$Rx = a_0 + c\,Ux,$$

where U is again a linear isometry.

Theorem 1.53. *Let $A \subset \mathbb{R}^n$. For the outer measure $\mathcal{H}^s$, $s \geq 0$, there holds:*

(a) $\mathcal{H}^s(A + x_0) = \mathcal{H}^s(A) \quad \forall x_0 \in \mathbb{R}^n$,

(b) $\mathcal{H}^s\big(U(A)\big) = \mathcal{H}^s(A)$ *for all linear isometries* $U\colon \mathbb{R}^n \to \mathbb{R}^n$,

(c) $\mathcal{H}^s\big(R(A)\big) = c^s\mathcal{H}^s(A)$, *if* $R\colon \mathbb{R}^n \to \mathbb{R}^n$ *is a similarity map, with scaling factor* $c > 0$.

Proof. The claims follow from the observation that $d\big(R(E)\big) = c\,d(E)$ $\forall E \subset \mathbb{R}^n$, where R is as in part (c). □

Let (X, d_1) and (Y, d_2) be metric spaces. Recall next that the mapping $f\colon X \to Y$ is *L-Lipschitz* (with a constant $L > 0$), if

$$d_2\big(f(x), f(y)\big) \leq L\,d_1(x, y)$$

for all $x, y \in X$. In the same way, a mapping $g\colon X \to Y$ is *L-bilipschitz*, if

$$\frac{1}{L}\,d_1(x, y) \leq d_2\big(f(x), f(y)\big) \leq L\,d_1(x, y)$$

for all $x, y \in X$. We observe that an L-bilipschitz mapping is always an injection because of the inequality on the left hand side.

Lemma 1.54. *Let (X, d_1) and (Y, d_2) be separable metric spaces.*

(i) If $f\colon X \to Y$ is L-Lipschitz, then

$$\mathcal{H}^s(fA) \leq L^s\mathcal{H}^s(A) \quad \forall A \subset X.$$

(ii) If $g\colon X \to Y$ is L-bilipschitz, then

$$\dim_{\mathcal{H}}(gA) = \dim_{\mathcal{H}}(A) \quad \forall A \subset X.$$

Proof. (i) We may suppose that $\mathcal{H}^s(A) < \infty$. Fix $\varepsilon > 0$, $\delta > 0$ and choose a δ-covering $\{E_j\}_{j=1}^{\infty}$ of A s.t.

$$\omega_s \sum_{j=1}^{\infty} \big(d(E_j)/2\big)^s \leq \mathcal{H}^s_\delta(A) + \varepsilon.$$

Then $\{f(E_j)\}_{j=1}^{\infty}$ is a $L\delta$-covering of fA and hence

$$\begin{aligned}\mathcal{H}^s_{L\delta}(fA) &\le \omega_s \sum_{j=1}^{\infty} \big(d\big(f(E_j)\big)/2\big)^s \\ &\le L^s \omega_s \sum_{j=1}^{\infty} \big(d(E_j)/2\big)^s \\ &\le L^s(\mathcal{H}^s_\delta(A) + \varepsilon).\end{aligned}$$

The claim follows by letting $\varepsilon \to 0$ and $\delta \to 0$.

(ii) Applying part (i) to the mapping $g^{-1}\colon g(A) \to X$ we obtain

$$L^{-s}\mathcal{H}^s(A) \le \mathcal{H}^s(gA).$$

Thus

$$L^{-s}\mathcal{H}^s(A) \le \mathcal{H}^s(gA) \le L^s\mathcal{H}^s(A),$$

which yields the claim.

□

We next construct sets with noninteger Hausdorff dimension. Recall the construction of the Cantor set from Real Analysis I. (We use slightly different notation and consider only a special case.)

Let $0 < \lambda < 1/2$. Denote $I_{0,1} = [0,1]$, $I_{1,1} = [0,\lambda]$ and $I_{1,2} = [1-\lambda, 1]$. In other words, $I_{1,1}$ and $I_{1,2}$ is obtained from $I_{0,1}$ by removing its middle interval with length $1-2\lambda$. Next we remove open interval of length $(1-2\lambda)\lambda$ from the middle of closed intervals $I_{1,i}$ and continue the process inductively. Suppose that the intervals $I_{n,i},\ i = 1,\dots,2^n$ of step n have been defined. Then the intervals $I_{n+1,j},\ j = 1,\dots,2^{n+1}$ of the step $(n+1)$ are obtained by removing an open interval of length $(1-2\lambda)\lambda^n$ from the middle of the intervals of step n. Thus

$$d(I_{n,i}) = \lambda^n, \quad \forall n \text{ and } \forall i = 1,\dots,2^n.$$

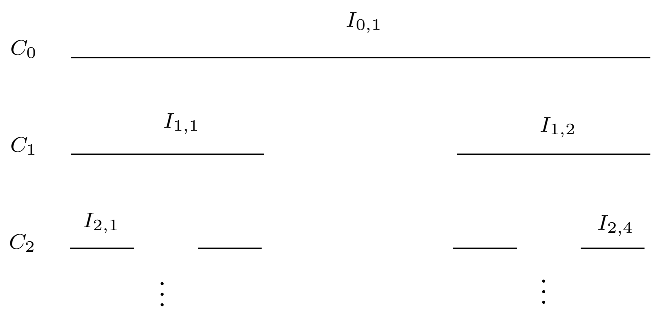
Denote

$$C_n(\lambda) = \bigcup_{i=1}^{2^n} I_{n,i}$$

("approximation of the nth step") and

$$C(\lambda) = \bigcap_{n=1}^{\infty} C_n(\lambda).$$

Then $C(\lambda)$ is compact, uncountable set, without interior points. Furthermore $C(\lambda)$ is "selfsimilar" and $m_1\big(C(\lambda)\big) = 0$. Cantor's 1/3-set $C(1/3)$ is a special case of this construction, recurrent in literature.



Theorem 1.55. *For all* $0 < \lambda < 1/2$

$$\dim_{\mathcal{H}} C(\lambda) = \frac{\log 2}{\log(1/\lambda)}.$$

In particular, $\dim_{\mathcal{H}} C(\lambda)$ *can attain all values in the interval* $(0,1)$.

Proof. It is enough to show that

$$2^{-1-s}\omega_s \le \mathcal{H}^s\big(C(\lambda)\big) \le 2^{-s}\omega_s, \tag{1.56}$$

if

$$s = \frac{\log 2}{\log(1/\lambda)}.$$

(i) We first give a *heuristic argument* for finding the exponent s: Clearly

$$C(\lambda) = C_1 \cup C_2,$$

where C_1 and C_2 are disjoint and similar to $C(\lambda)$ with the scaling factor λ. If $C(\lambda)$ *would* satisfy (1.56), then by part (c) of Theorem 1.53

$$\begin{aligned}\mathcal{H}^s\big(C(\lambda)\big) &= \mathcal{H}^s(C_1) + \mathcal{H}^s(C_2)\\ &= 2\lambda^s \mathcal{H}^s\big(C(\lambda)\big).\end{aligned}$$

Thus

$$1 = 2\lambda^s,$$

Solving this for s yields

$$s = \frac{\log 2}{\log(1/\lambda)}.$$

(ii) A rigorous proof of (1.56): If $\delta > 0$ is given, then choose $n \in \mathbb{N}$ so large that $\lambda^n < \delta$. Then $\{I_{n,i}\}_{i=1}^{2^n}$ is a δ-covering of $C(\lambda)$ and, furthermore,

$$\mathcal{H}^s_\delta\big(C(\lambda)\big) \le \omega_s \sum_{i=1}^{2^n} (\lambda^n/2)^s = 2^{-s}\omega_s \sum_{i=1}^{2^n} (\lambda^n)^s = 2^{-s}\omega_s \sum_{i=1}^{2^n} (1/2)^n = 2^{-s}\omega_s.$$

Thus

$$\mathcal{H}^s\big(C(\lambda)\big) \le 2^{-s}\omega_s.$$

We give a proof for the lower bound (1.56) only in the special case $\lambda = 1/3$. The general case $\lambda \in (0,1/2)$ would not bring any essential changes to the proof. Suppose that $\{E_j\}_{j=1}^\infty$ is a δ-covering of $C(1/3)$ such that

$$\omega_s \sum_{j=1}^\infty \big(d(E_j)/2\big)^s \le \mathcal{H}^s\big(C(1/3)\big) + \delta, \quad s = \frac{\log 2}{\log 3}.$$

For each j choose a closed interval I_j $(= [a,b])$ s.t. $E_j \subset \operatorname{int} I_j$ $(=]a,b[)$ and $d(I_j) < (1+\delta)d(E_j)$. Then $\{\operatorname{int} I_j\}_{j=1}^\infty$ is an open covering of $C(1/3)$, and hence by the compactness of $C(1/3)$ we can choose a finite subcover. By relabelling the intervals I_j, we may suppose that

$$C(1/3) \subset \bigcup_{j=1}^m I_j$$

and

$$\begin{aligned}\mathcal{H}^s\big(C(1/3)\big) + \delta &\geq \omega_s \sum_{j=1}^{\infty} \big(d(E_j)/2\big)^s \\ &\geq \omega_s 2^{-s}(1+\delta)^{-s} \sum_{j=1}^{m} d(I_j)^s.\end{aligned}$$

To prove the lower bound (1.56) it is enough to prove that

$$\sum_{j=1}^{m} d(I_j)^s \geq \frac{1}{2}, \tag{1.57}$$

if $\{I_j\}_{j=1}^m$ is a covering of $C(1/3)$ with finitely many closed intervals I_j. For each j choose $k = k(j) \in \mathbb{N}$, with

$$3^{-(k+1)} \leq d(I_j) < 3^{-k}. \tag{1.58}$$

Let k_0 be the largest one of the numbers $k(j), j = 1, \dots, m$. On the basis of the construction and the choice of the number $k = k(j)$, each I_j can intersect only one of the intervals $I_{k,i}$ of step k. Therefore I_j intersects at most $2^{k_0 - k(j)}$ of the intervals $I_{k_0,i}$. Thus the number of such intervals of step k_0 is at most

$$\sum_{j=1}^{m} 2^{k_0 - k(j)}.$$

On the other hand, there are 2^{k_0} intervals of step k_0. Every one of these contains points of $C(1/3)$ and $C(1/3) \subset \bigcup_{j=1}^m I_j$, and hence

$$2^{k_0} \leq \sum_{j=1}^{m} 2^{k_0 - k(j)}.$$

Now we can compute

$$\begin{aligned}2^{k_0} &\leq \sum_{j=1}^{m} 2^{k_0 - k(j)} = 2^{k_0} \sum_{j=1}^{m} 2^{-k(j)} \\ &= 2^{k_0} \sum_{j=1}^{m} (3^{-k(j)})^s \\ &\leq 2^{k_0} \sum_{j=1}^{m} \big(3d(I_j)\big)^s.\end{aligned}$$

Simplification yields

$$\sum_{j=1}^{m} d(I_j)^s \geq 3^{-s} = 1/2.$$

□

Remark 1.59. Refining the above argument one can show that

$$\mathcal{H}^s\big(C(\lambda)\big) = 1, \;\; s = \frac{\log 2}{\log(1/\lambda)},$$

(cf. Falconer, K. J.: The geometry of fractal sets, Cambridge University Press, 1985, pages 14-15).

1.60 Riesz representation theorem

Recall the following general form of the Riesz representation theorem. Let X be a locally compact, separable metric space, and let H be a finite dimensional Hilbert space with the inner product $(\cdot,\cdot)$. Denote by $C_0(X,H)$ the space of all continuous mappings $X \to H$ with compact support. If $L\colon C_0(X,H) \to \mathbb{R}$ is a linear functional such that

$$\sup\{L(f)\colon f \in C_0(X,H), |f| \le 1, \operatorname{supp} f \subset K\} < \infty$$

for all compact $K \subset X$, there exist a Radon measure μ and a μ-measurable $\nu\colon X \to H$ such that $|\nu(x)| = 1$ for ν-a.e. $x \in X$ and

$$L(f) = \int_X (f,\nu)\,d\mu$$

for every $f \in C_0(X,H)$.

See, for example, [Si, Theorem 4.1]. We will consider the special case $X = \mathbb{R}^n$ and $H = \mathbb{R}$ in the home work classes.

Definition 1.61. A mapping $\Lambda\colon C_0(\mathbb{R}^n,\mathbb{R}) \to \mathbb{R}$ is a *positive linear functional* if

(i) $\Lambda(\alpha f_1 + \beta f_2) = \alpha\Lambda(f_1) + \beta\Lambda(f_2)$ for all $f_1, f_2 \in C_0(\mathbb{R}^n,\mathbb{R})$ and all $\alpha,\beta \in \mathbb{R}$.

(ii) $\Lambda(f) \ge 0$ for all $f \in C_0(\mathbb{R}^n,\mathbb{R})$, with $f(x) \ge 0\ \forall x \in \mathbb{R}^n$.

Theorem 1.62 (Riesz representation theorem)**.** *Let $\Lambda\colon C_0(\mathbb{R}^n,\mathbb{R}) \to \mathbb{R}$ be a positive linear functional. Then there exists a unique Radon measure μ, more precisely, a measure space $(\mathbb{R}^n, \operatorname{Bor}(\mathbb{R}^n), \mu)$, such that*

$$\Lambda(f) = \int_{\mathbb{R}^n} f(x)\,d\mu(x)$$

for all $f \in C_0(\mathbb{R}^n,\mathbb{R})$.

The proof of Riesz representation theorem is based on an auxiliary result.

Lemma 1.63. *(a) Let $V \subset \mathbb{R}^n$ be open and $K \subset V$ compact. Then there exists $f \in C_0(\mathbb{R}^n,\mathbb{R})$ such that*

$$\operatorname{supp}(f) \subset V \quad \textit{and} \quad \chi_K(x) \le f(x) \le 1 \quad \forall x \in \mathbb{R}^n.$$

(b) Let $V_j \subset \mathbb{R}^n$, $j = 1,\dots,m$, be open and $K \subset \bigcup_{j=1}^m V_j$ compact. Then there exist functions $h_j \in C_0(\mathbb{R}^n,\mathbb{R})$, with

$$0 \le h_j \le 1, \quad \operatorname{supp}(h_j) \subset V_j \quad \textit{and} \quad \chi_K \le \sum_{j=1}^m h_j \le 1.$$

1.64 Weak convergence of measures

Definition 1.65. Let μ_k, $k \in \mathbb{N}$, be Radon measures in $\mathbb{R}^n$. We say that a sequence (μ_k) *converges weakly* to a Radon measure a μ, if

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} f\,d\mu_k = \int_{\mathbb{R}^n} f\,d\mu$$

all $f \in C_0(\mathbb{R}^n,\mathbb{R})$. This is denoted by $\mu_k \rightharpoonup \mu$ or $\mu_k \xrightarrow{\text{w}} \mu$.

Example 1.66. (i) Let δ_x be the Dirac measure at $x \in \mathbb{R}$. Then $\delta_k \rightharpoonup 0$.

(ii) Let $\mu_k = \frac{1}{k}\big(\delta_{1/k} + \delta_{2/k} + \cdots + \delta_{k/k}\big)$. Then for all $f \in C_0(\mathbb{R}, \mathbb{R})$

$$\int_{\mathbb{R}} f \, d\mu_k = \sum_{j=1}^{k} \frac{1}{k} f(j/k) \xrightarrow{k\to\infty} \int_0^1 f(x)\, dx,$$

because the sums are Riemann sums of the function f on $[0,1]$. Thus $\mu_k \rightharpoonup m_1 \llcorner [0,1]$.

Easy examples show that it is not always true that $\mu_k(A) \to \mu(A)$, if $\mu_k \rightharpoonup \mu$. However, there holds:

Theorem 1.67. *Let μ, μ_k, $k \in \mathbb{N}$, be Radon measures in $\mathbb{R}^n$ such that $\mu_k \rightharpoonup \mu$. Then*

(a) $\limsup_{k\to\infty} \mu_k(K) \le \mu(K)$ *if $K \subset \mathbb{R}^n$ is compact,*

(b) $\liminf_{k\to\infty} \mu_k(V) \ge \mu(V)$ *if $V \subset \mathbb{R}^n$ is open.*

1.68 Compactness of measures

The weak convergence of measures is not merely natural but also a very useful notion. The families of bounded Radon measures are sequentially compact. In many cases this is the only way to construct measures (as limiting measures of weakly convergent sequences).

Theorem 1.69. *Let (μ_k) be a sequence of Radon measures in $\mathbb{R}^n$ with*

$$\sup_k \mu_k(K) < \infty$$

for all compact $K \subset \mathbb{R}^n$. Then there exists a subsequence (μ_{k_j}) and a Radon measure μ with

$$\mu_{k_j} \rightharpoonup \mu.$$

The proof of of this theorem will be discussed in the home work classes.

2 Lipschitz mappings and rectifiable sets

2.1 Extension of Lipschitz mappings

Next we present a useful extension result of Lipschitz mappings.

Theorem 2.2 (McShane-Whitney extension theorem)**.** *Let X be a metric space, $A \subset X$, and $f\colon A \to \mathbb{R}$ L-Lipschitz. Then there exists an L-Lipschitz function $F\colon X \to \mathbb{R}$ such that $F|A = f$.*

Proof. For every $a \in A$ we define an L-Lipschitz function $f^a\colon X \to \mathbb{R}$

$$f^a(x) = f(a) + L|a - x|, \quad x \in X.$$

The function F is then defined by setting

$$F(x) = \inf_{a\in A} f^a(x), \quad x \in X.$$

Clearly $F(x) < \infty$ $\forall x \in X$. By fixing $a_0 \in A$ we see that

$$\begin{aligned} f(a) + L|a - x| &\geq f(a) + L|a - a_0| - L|a_0 - x| \\ &\geq f(a_0) - L|a_0 - x|. \end{aligned}$$

Hence $F(x) > -\infty$ for all $x \in X$. Since every f^a is L-Lipschitz and $F(x) > -\infty$ for all $x \in X$, F is L-Lipschitz. Moreover, for every $x \in A$

$$F(x) \leq f^x(x) = f(x) \leq f(y) + L|x - y| = f^y(x) \quad \forall y \in A,$$

and hence $F|A = f$. □

Corollary 2.3. *Let X be a metric space, $A \subset X$, and $f\colon A \to \mathbb{R}^n$ L-Lipschitz. Then there exists a $\sqrt{n}L$-Lipschitz mapping $F\colon X \to \mathbb{R}^n$ such that $F|A = f$.*

Proof. Apply Theorem 2.2 to the coordinate functions of f. □

Remark 2.4.

1. Theorem 2.2 holds (as such) in the case $X \subset \mathbb{R}^m$, $f\colon X \to \mathbb{R}^n$, but the proof is much harder. This is so called Kirzbraun's theorem.
2. It is a topic of quite active current research to study which pairs of metric spaces X, Y have a Lipschitz extension property (i.e. for every $A \subset X$ every Lipschitz mapping $f\colon A$ has a Lipschitz extension $F\colon X \to Y$).

2.5 Rademacher's theorem

According to Rademacher's theorem a Lipschitz mapping $\mathbb{R}^n \to \mathbb{R}^m$ is differentiable m_n-a.e. Let us first recall the following definition.

Definition 2.6. A mapping $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is *differentiable at* $x \in \mathbb{R}^n$ if there exits a linear mapping $L\colon \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{y \to x} \frac{|f(y) - f(x) - L(y - x)|}{|y - x|} = 0$$

or, equivalently,

$$f(y) = f(x) + L(y - x) + o(|y - x|) \text{ as } y \to x.$$

If such L exists, it is unique and we denote it by $Df(x)$ and call it the *derivative* of f at x or the *differential* of f at x.

Theorem 2.7 (Rademacher's theorem)**.** *Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be locally Lipschitz, i.e. for each compact $K \subset \mathbb{R}^n$ there exist a constant $L_K < \infty$ such that*

$$|f(x) - f(y)| \leq L_K|x - y| \quad \forall x, y \in K.$$

Then f is differentiable m_n-a.e. in $\mathbb{R}^n$.

The proof will be discussed in home work sessions.

2.8 Linear maps and Jacobians

Let us start with the following fact: Suppose that $L\colon \mathbb{R}^n \to \mathbb{R}^n$ is linear. Then

$$m_n^*(LA) = |\det L| m_n^*(A)$$

for every $A \subset \mathbb{R}^n$. We will not prove this formula (think of the special case $Lx = (c_1x_1, c_2x_2, \dots, c_nx_n)$, where $c_1, \dots, c_n \in \mathbb{R}$). We want to have a counterpart of this "area formula" in case $L\colon \mathbb{R}^n \to \mathbb{R}^m$ is linear. Towards this end, let us first recall the following notions related to linear algebra (without proofs).

Definition 2.9. (i) A linear map $O\colon \mathbb{R}^n \to \mathbb{R}^m$ is *orthogonal* if

$$Ox \cdot Oy = x \cdot y \quad \forall x, y \in \mathbb{R}^n.$$

(ii) A linear map $S\colon \mathbb{R}^n \to \mathbb{R}^n$ is *symmetric* if

$$x \cdot Sy = Sx \cdot y \quad \forall x, y \in \mathbb{R}^n.$$

(iii) A linear map $D\colon \mathbb{R}^n \to \mathbb{R}^n$ is *diagonal* if there are constants $d_1, \dots, d_n$ such that

$$Dx = (d_1x_1, \dots, d_nx_n) \quad \forall x = (x_1, \dots, x_n) \in \mathbb{R}^n.$$

(iv) The *adjoint* of a linear map $L\colon \mathbb{R}^n \to \mathbb{R}^m$ is the linear map $L^*\colon \mathbb{R}^m \to \mathbb{R}^n$ defined by

$$x \cdot L^*y = (Lx) \cdot y \quad \forall x \in \mathbb{R}^n, y \in \mathbb{R}^m.$$

Theorem 2.10. *(i)* $L^{**} = L$.

(ii) $(AB)^* = B^*A^*$.

(iii) $O^* = O^{-1}$ *if* $O\colon \mathbb{R}^n \to \mathbb{R}^n$ *is orthogonal.*

(iv) $S^* = S$ *if* $S\colon \mathbb{R}^n \to \mathbb{R}^n$ *is symmetric.*

(v) For every symmetric map $S\colon \mathbb{R}^n \to \mathbb{R}^n$ *there exist an orthogonal map* $O\colon \mathbb{R}^n \to \mathbb{R}^n$ *and a diagonal map* $D\colon \mathbb{R}^n \to \mathbb{R}^n$ *such that*

$$S = ODO^{-1}.$$

(vi) If $O\colon \mathbb{R}^n \to \mathbb{R}^m$ *is orthogonal, then* $n \le m$ *and*

$$O^*O = \mathrm{id} \ \text{ in } \mathbb{R}^n,$$
$$OO^* = \mathrm{id} \ \text{ in } O\mathbb{R}^n.$$

Theorem 2.11 (Polar decomposition)**.** *Let* $L\colon \mathbb{R}^n \to \mathbb{R}^m$ *be a linear mapping.*

(i) If $n \le m$*, there exists a symmetric map* $S\colon \mathbb{R}^n \to \mathbb{R}^n$ *and an orthogonal map* $O\colon \mathbb{R}^n \to \mathbb{R}^m$ *such that*

$$L = OS.$$

(ii) If $n \ge m$*, there a symmetric map* $S\colon \mathbb{R}^m \to \mathbb{R}^m$ *and an orthogonal map* $O\colon \mathbb{R}^m \to \mathbb{R}^n$ *such that*

$$L = SO^*.$$

For the proof, see e.g. [EG]. We are now ready to define the *Jacobian* of a linear map $L\colon \mathbb{R}^n \to \mathbb{R}^m$.

Definition 2.12. Let $L\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear mapping.

(i) If $n \le m$, let $L = OS$ be as above and define the Jacobian of L as

$$[\![L]\!] = |\det S|.$$

(ii) If $n \ge m$, let $L = SO^*$ be as above and define the Jacobian of L as

$$[\![L]\!] = |\det S|.$$

Theorem 2.13. *(i) If $n \le m$, then $[\![L]\!]^2 = \det(L^*L)$.*

(ii) If $n \ge m$, then $[\![L]\!]^2 = \det(LL^*)$.

(iii) $[\![L]\!] = [\![L^*]\!]$.

Remark 2.14. The Jacobian $[\![L]\!]$ is well-defined since it is independent of the choices of S and O by Theorem 2.13.

2.15 Jacobians of Lipschitz mappings

Let $f = (f_1, \dots, f_m)\colon \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz mapping. By Rademacher's theorem, f is differentiable at m_n-a.e. $x \in \mathbb{R}^n$. Hence the derivative $Df(x)$ exists and can be expressed as a matrix

$$Df(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

at m_n-a.e. $x \in \mathbb{R}^n$.

Definition 2.16. The *Jacobian* of f at a point x, where f is differentiable, is

$$J_f(x) = [\![Df(x)]\!].$$

2.17 The area formula

Some details will be discussed in the home work classes.

In this subsection we assume that $n \le m$ and that $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz.

Lemma 2.18 (Area formula for linear maps)**.** *Let $L\colon \mathbb{R}^n \to \mathbb{R}^m$, $n \le m$, be a linear map. Then*

$$\mathcal{H}^n(LA) = [\![L]\!] m_n^*(A) \quad \forall A \subset \mathbb{R}^n.$$

Note that we have defined $\mathcal{H}^n$ so that $\mathcal{H}^n = m_n^*$ in $\mathbb{R}^n$.

Lemma 2.19. *Let $A \subset \mathbb{R}^n$ be Lebesgue measurable. Then*

(i) fA is $\mathcal{H}^n$-measurable,

(ii) the mapping $y \mapsto \mathcal{H}^0\big(A \cap f^{-1}(y)\big)$ is $\mathcal{H}^n$-measurable, and

(iii)

$$\int_{\mathbb{R}^m} \mathcal{H}^0\big(A \cap f^{-1}(y)\big) d\mathcal{H}^n(y) \le \big(\mathrm{Lip}(f)\big)^n m_n(A).$$

Lemma 2.20. *Let $t > 1$ and $B = \{x \in \mathbb{R}^n\colon$ the derivative $Df(x)$ exists and $J_f(x) > 0\}$. Then there exists a countable collection $E_k \in \mathrm{Bor}(\mathbb{R}^n), k \in \mathbb{N}$, such that*

(i)

$$B = \bigcup_{k=1}^{\infty} E_k,$$

(ii) $f|E_k$ is one-to-one,

(iii) for every $k \in \mathbb{N}$ there exists a symmetric automorphism $T_k\colon \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\mathrm{Lip}\big((f|E_k) \circ T_k^{-1}\big) \le t,$$
$$\mathrm{Lip}\big(T_k \circ (f|E_k)^{-1}\big) \le t,$$
$$t^{-n}|\det T_k| \le J_{f|E_k} \le t^n|\det T_k|.$$

The message of the lemma is that f can be locally approximated by a symmetric automorphism as closely as we wish.

Theorem 2.21 (The area formula)**.** *Let $f\colon \mathbb{R}^n \to \mathbb{R}^m, n \le m$, be a Lipschitz mapping. Then for every m_n-measurable set $A \subset \mathbb{R}^n$*

$$\int_A J_f(x) dm_n(x) = \int_{\mathbb{R}^m} \mathcal{H}^0\big(A \cap f^{-1}(y)\big) d\mathcal{H}^n(y).$$

Corollary 2.22 (Change of variables)**.** *Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz map, $n \le m$. Then for each m_n-integrable $g\colon \mathbb{R}^n \to \mathbb{R}$*

$$\int_{\mathbb{R}^n} g(x) J_f(x) dm_n(x) = \int_{\mathbb{R}^m} \left(\sum_{x \in f^{-1}(y)} g(x) \right) d\mathcal{H}^n(y).$$

As an application let us consider the surface area (measure) of the graph of a Lipschitz function $g\colon \mathbb{R}^n \to \mathbb{R}$. Define $f\colon \mathbb{R}^n \to \mathbb{R}^{n+1}$, $f(x) = \Big(x, g(x)\Big)$. Then

$$Df = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & \vdots \\ \vdots & 0 & & & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} & \cdots & \cdots & \frac{\partial g}{\partial x_n} \end{pmatrix} = \begin{pmatrix} I_n \\ \nabla g \end{pmatrix}$$

and $J_f^2 = 1 + |\nabla g|^2$. For each open $U \subset \mathbb{R}^n$, the graph of g over U is

$$\Gamma = \Gamma_{g,U} = \{(x, g(x))\colon x \in U\}$$

and

$$\mathcal{H}^n(\Gamma) = \int_U \sqrt{1 + |\nabla g|^2} dm_n(x).$$

2.23 The co-area formula

In this subsection we assume that $n \geq m$. Some details will be discussed in home work classes.

Let us start with linear maps $L\colon \mathbb{R}^n \to \mathbb{R}^m,\ n \geq m$. Consider first the special case where $L\colon \mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}^m$ is the orthogonal projection onto $\mathbb{R}^m$,

$$L(x_1, \dots, x_m, x_{m+1}, \dots, x_{m+k}) = (x_1, \dots, x_m).$$

Then for each $y \in \mathbb{R}^m$ the preimage $L^{-1}(y)$ is an affine $(n-m)$-dimensional subspace. The preimages $L^{-1}(y),\ y \in \mathbb{R}^m$, decompose $\mathbb{R}^n$ into parallel $(n-m)$-dimensional slices. By Fubini's theorem

$$\int_{\mathbb{R}^m} \mathcal{H}^{n-m}\big(L^{-1}(y) \cap A\big)\, dm_m(y) = \mathcal{H}^n(A) = m_n(A)$$

whenever $A \subset \mathbb{R}^n$ is Lebesgue measurable. For a general linear map $L\colon \mathbb{R}^n \to \mathbb{R}^m,\ n \geq m$, we have the following.

Lemma 2.24. *Let $L\colon \mathbb{R}^n \to \mathbb{R}^m,\ n \geq m$, be a linear mapping and $A \subset \mathbb{R}^n$ Lebesgue measurable. Then*

(i) the mapping $y \mapsto \mathcal{H}^{n-m}\big(A \cap L^{-1}(y)\big)$ is m_m-measurable, and

(ii)
$$\int_{\mathbb{R}^m} \mathcal{H}^{n-m}\big(A \cap L^{-1}(y)\big)\, dm_m(y) = [\![L]\!] m_n(A).$$

Similarly to the case of area formula, we have:

Lemma 2.25. *Let $f\colon \mathbb{R}^n \to \mathbb{R}^m,\ n \geq m$, be a Lipschitz mapping and $A \subset \mathbb{R}^n$ Lebesgue measurable. Then*

(i) fA is m_m-measurable,

(ii) $A \cap f^{-1}(y)$ is $\mathcal{H}^{n-m}$-measurable for m_m-a.e. $y \in \mathbb{R}^m$,

(iii) the mapping $y \mapsto \mathcal{H}^{n-m}\big(A \cap f^{-1}(y)\big)$ is m_m-measurable, and

(iv)
$$\int_{\mathbb{R}^m} \mathcal{H}^{n-m}\big(A \cap f^{-1}(y)\big)\, dm_m(y) \leq \frac{\omega_{n-m}\omega_m}{\omega_n} (\operatorname{Lip} f)^m m_n(A).$$

Lemma 2.26. *Let $t > 1$ and $h\colon \mathbb{R}^n \to \mathbb{R}^n$ Lipschitz. Let*

$$B = \{x \in \mathbb{R}^n \colon Dh(x) \text{ exists and } J_h(x) > 0\}.$$

Then there exists a countable collection $D_k \in \operatorname{Bor}(\mathbb{R}^n), k \in \mathbb{N}$, such that

(i) $m_n\big(B \setminus \cup_k D_k\big) = 0$,

(ii) $h|D_k$ is one-to-one, and

(iii) for each k there exists a symmetric automorphism $S_k\colon \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\operatorname{Lip}\big(S_k^{-1} \circ (h|D_k)\big) \leq t,$$
$$\operatorname{Lip}\big((h|D_k)^{-1} \circ S_k\big) \leq t,$$
$$t^{-n}|\det S_k| \leq J_{h|E_k} \leq t^n |\det S_k|.$$

Note that above both the domain and the target of h is $\mathbb{R}^n$. For the proof of Lemma 2.26 we apply Lemma 2.20 to find sets E_k such that each $h|E_k$ is one-to-one and then we apply Lemma 2.20 again to $(h|E_k)^{-1}$ in hE_k.

Theorem 2.27 (The co-area formula). *Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$, $n \geq m$, be a Lipschitz map. Then for each m_n-measurable set $A \subset \mathbb{R}^n$*

$$\int_A J_f(x)dm_n(x) = \int_{\mathbb{R}^m} \mathcal{H}^{n-m}\big(A \cap f^{-1}(y)\big)dm_m(y).$$

Corollary 2.28 (Change of variables). *Let $f\colon \mathbb{R}^n \to \mathbb{R}^m, n \geq m$, be a Lipschitz mapping. Then for each m_n-integrable function $g\colon \mathbb{R}^n \to \mathbb{R}$, $g|f^{-1}(y)$ is $\mathcal{H}^{n-m}$-integrable for m_m-a.e. $y \in \mathbb{R}^m$ and*

$$\int_{\mathbb{R}^n} g(x)J_f(x)dm_n(x) = \int_{\mathbb{R}^m} \left(\int_{f^{-1}(y)} gd\mathcal{H}^{n-m}\right) dm_m(y).$$

As an application we consider level sets of a Lipschitz function $f\colon \mathbb{R}^n \to \mathbb{R}$. Then $J_f = |\nabla f|$ and hence

$$\int_{\mathbb{R}^n} |\nabla f|dm_n = \int_{-\infty}^{\infty} \mathcal{H}^{n-1}\big(\{f = t\}\big)dt.$$

2.29 Rectifiable sets

Let us start with the following two examples

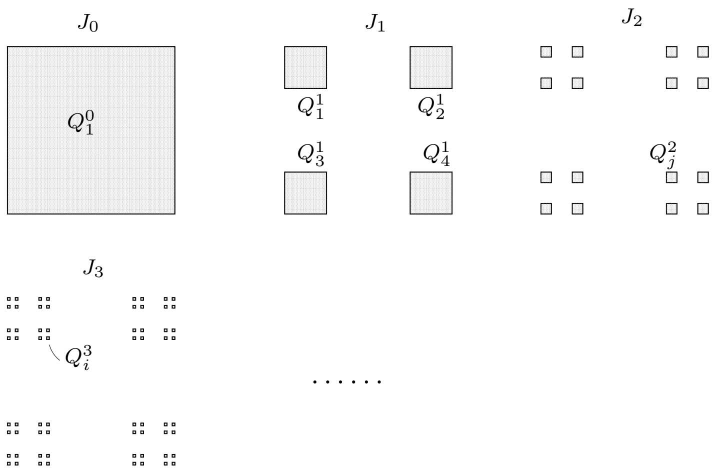
Example 2.30. Let $J_0 = Q_1^0 = [0,1]^2$ be the closed unit square of the plane and let J_1 be the union of four closed squares Q_i^1, $i = 1, \dots, 4$, in its corners, each with edge length $1/4$. In the next step each of the four squares Q_i^1, $i = 1, \dots, 4$ of J_1, will be replaced with four corner squares Q_j^2, $j = 1, \dots, 16$, each with edge length $1/16$. Continuing in this way in the step n we have 4^n squares Q_j^n, $j = 1, \dots, 4^n$, each with egde length 4^{-n}. Let

$$J_n = \bigcup_{j=1}^{4^n} Q_j^n$$

and

$$J = \bigcap_{n=0}^{\infty} J_n.$$

Then J is a set of Cantor type. In fact, $J = C(1/4) \times C(1/4)$.



What is the Hausdorff dimension $\dim_{\mathcal{H}} J$ of J? We find a suitable candidate for $\dim_{\mathcal{H}} J$ by using similarities. Observe that

$$J = \bigsqcup_{i=1}^{4} \tilde{J}_i,$$

where $\tilde{J}_i$ is similar to J with the scaling factor $1/4$, and hence $\mathcal{H}^s(\tilde{J}_i) = (1/4)^s \mathcal{H}^s(J)$ and further

$$\mathcal{H}^s(J) = 4(1/4)^s \mathcal{H}^s(J).$$

If J is an s-set (i.e. $0 < \mathcal{H}^s(J) < \infty$), then we get from above that

$$4(1/4)^s = 1$$

which gives $s = 1$. Let us give some further details. Fix $\delta > 0$. Observe first that $d(Q_j^n) = \sqrt{2}/4^n$. If $n \in \mathbb{N}$ is so large that $\sqrt{2}/4^n \le \delta$, then $\{Q_j^n\}_{j=1}^{4^n}$ is a δ-covering of J and thus

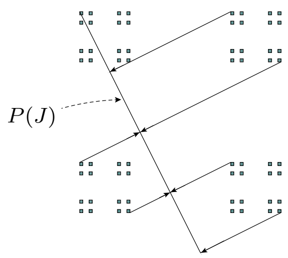
$$\mathcal{H}^1_\delta(J) \le \sum_{j=1}^{4^n} d(Q_j^n) \le 4^n \sqrt{2}/4^n = \sqrt{2}.$$

Therefore $\mathcal{H}^1(J) \le \sqrt{2} < \infty$. By an argument similar to that in the proof of Theorem 1.55 one can show that $\mathcal{H}^1(J) > 0$. Thus

$$\dim_{\mathcal{H}}(J) = 1 \quad \text{and} \quad 0 < \mathcal{H}^1(J) < \infty,$$

in other words J is a 1-set. However, its geometric structure is very different from that of a rectifiable curve.

Remark.: A positive lower bound can also be found by using an orthogonal projection $P\colon \mathbb{R}^2 \to S$ onto a line S with slope -2. Then the image set $P(J)$ is a segment with length $3/\sqrt{5}$. Because the projection P is 1-Lipschitz, then $\mathcal{H}^1(J) \ge \mathcal{H}^1(P(J)) = 3/\sqrt{5}$. In fact, it can be shown that $\mathcal{H}^1(J) = \sqrt{2}$.



It was pointed out above that $J = C(1/4) \times C(1/4)$. Observe that $\dim_{\mathcal{H}}(J) = 1 = 2\log 2/\log 4 = \dim_{\mathcal{H}} C(1/4) + \dim_{\mathcal{H}} C(1/4)$.

Example 2.31. Let $q_1, q_2, \ldots$ be those points of the closed unit disk $\bar{D} = \{x \in \mathbb{R}^2 \colon |x| \le 1\}$ whose both coordinates are rational numbers. These points form a countable dense subset of $\bar{D}$. Let

$$E = \bigcup_{j=1}^{\infty} S_j,$$

where

$$S_j = \{x \in \mathbb{R}^2 \colon |x - q_j| = 2^{-j}\}.$$

Now

$$0 < \mathcal{H}^1(E) \le \sum_{j=1}^{\infty} 2\pi\, 2^{-j} = 2\pi < \infty.$$

Thus E is a 1-set and $\dim_{\mathcal{H}} E = 1$. However, E is dense in $\bar{D}$, $\bar{E} \cap \bar{D} = \bar{D}$, and therefore E is "very big". In which sense does E resemble a rectifiable arc?

These and other similar examples raise several questions:

- In which sense the Cantor-set of Example 2.30 is different from a rectifiable arc?
- What kind of set is a general 1-set? Is there a way to distinguish between "Cantor-type" and "rectifiable" parts and how these parts could be defined?
- Rectifiable arcs have tangent lines a.e. Does this property have a counterpart for sets such as in Example 2.31?

Recall *Lebesgue's density theorem* from Real Analysis I: If $E \in \mathrm{Leb}(\mathbb{R}^n)$, then

$$\lim_{r \to 0+} \frac{m_n(E \cap B(x,r))}{m_n(B(x,r))} = 1$$

for a.e. $x \in E$. It is a natural question whether Hausdorff measures have some similar properties. Recall that we defined the Hausdorff measure so that $\mathcal{H}^n(\bar{B}(x,1)) = \omega_n$ for $B(x,r) \subset \mathbb{R}^n$. Keeping this in mind we define:

Definition 2.32. Let $0 \le s < \infty$, $A \subset \mathbb{R}^n$ and $a \in \mathbb{R}^n$. The *upper* and *lower s-densities of A at the point a* are

$$\Theta^{*s}(A,a) = \limsup_{r \to 0+} \frac{\mathcal{H}^s(A \cap \bar{B}(a,r))}{\omega_s r^s},$$
$$\Theta^{s}_{*}(A,a) = \liminf_{r \to 0+} \frac{\mathcal{H}^s(A \cap \bar{B}(a,r))}{\omega_s r^s}.$$

If $\Theta^s_*(A,a) = \Theta^{*s}(A,a)$, then this value is called the *(s-dimensional) density of A at a* and denoted by $\Theta^s(A,a)$.

We study densities using covering theorems. Recall from Real Analysis I the following basic covering theorem and the notion of a Vitali covering of a set. If B is an open (closed) ball centered at x with radius r, then $5B$ is an open (closed) ball centered at x with radius $5r$.

Theorem 2.33 (Basic covering theorem)**.** *Let $\mathcal{F}$ be an arbitrary family of balls of $\mathbb{R}^n$ s.t.*

$$D = \sup\{d(B) \colon B \in \mathcal{F}\} < \infty.$$

Then there exists a countable (possibly finite) family $\mathcal{G} \subset \mathcal{F}$ s.t.

$$B_i \cap B_j = \emptyset \quad \forall B_i, B_j \in \mathcal{G},\ B_i \ne B_j, \quad \textit{i.e. the balls of } \mathcal{G} \textit{ are disjoint; and}$$

$$\bigcup_{B \in \mathcal{F}} B \subset \bigcup_{B \in \mathcal{G}} 5B\,.$$

Definition 2.34. Let $\mathcal{V}$ be a family of balls in $\mathbb{R}^n$. We say that $\mathcal{V}$ is a *Vitali covering* of a set $E \subset \mathbb{R}^n$ if for every $x \in E$ and every $\varepsilon > 0$ there exists $B \in \mathcal{V}$ s.t. $x \in B$ and $d(B) < \varepsilon$. The family $\mathcal{V}$ is a closed (open) Vitali covering if every $B \in \mathcal{V}$ is closed (open) ball.

Theorem 2.35 (Vitali's covering theorem for Hausdorff measures)**.** *Let $0 < s < \infty$ and let $\mathcal{V}$ be a closed Vitali covering of a set $E \subset \mathbb{R}^n$. Then there exists a countable family of disjoint balls $B_i \in \mathcal{V}$ s.t. either*

$$\sum_{i=1} d(B_i)^s = \infty \tag{2.36}$$

or

$$\mathcal{H}^s\big(E \setminus \bigcup_{i=1} B_i\big) = 0. \tag{2.37}$$

Remark 2.38. Applying so called Besicovitch's covering theorem we obtain a counterpart of Theorem 2.35 for a general Radon-measure of $\mathbb{R}^n$ (see Theorems **??**, **??**).

Proof. We may suppose that $0 < d(B) < 1$ for all $B \in \mathcal{V}$. We choose the balls inductively: Let $B_1 \in \mathcal{V}$ be arbitrary. Suppose that disjoint balls $B_1, \ldots, B_m \in \mathcal{V}$ have been chosen. Let

$$d_m = \sup\{d(B)\colon B \in \mathcal{V},\ B \cap B_i = \emptyset \ \forall 1 \le i \le m\}.$$

If $d_m = 0$, then

$$E \subset \bigcup_{i=1}^m B_i,$$

and the claim is proven ((2.37) holds). Indeed, if there exists $x \in E \setminus \cup_{i=1}^m B_i$, then

$$\operatorname{dist}(x, \cup_{i=1}^m B_i) > 0,$$

because $\cup_{i=1}^m B_i$ is compact. Because $\mathcal{V}$ is a Vitali covering of E, there would exist $B \in \mathcal{V}$ s.t. $x \in B$ and $B \cap \cup_{i=1}^m B_i = \emptyset$ and therefore $d_m > 0$.

If $d_m > 0$, then choose $B_{m+1} \in \mathcal{V}$ such that $d(B_{m+1}) > d_m/2$. If this selection process will not end for any m, we obtain disjoint balls $\{B_i\}_{i=1}^\infty \subset \mathcal{V}$. Therefore we must show: If

$$\sum_{i=1}^\infty d(B_i)^s < \infty,$$

then the condition (2.37) holds. For this purpose we show first that

$$E \setminus \bigcup_{i=1}^k B_i \subset \bigcup_{j=k+1}^\infty 5B_j \quad \forall k \in \mathbb{N}. \tag{2.39}$$

Indeed, if

$$x \in E \setminus \bigcup_{i=1}^k B_i,$$

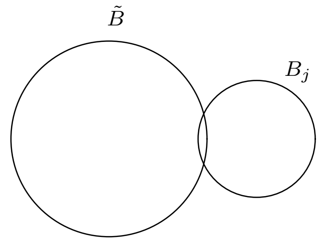
then $x \in \tilde{B} \in \mathcal{V}$, where $\tilde{B} \cap B_i = \emptyset$ for all $1 \le j \le k$. Because $d(B_m) \to 0$ as $m \to \infty$, then $d(\tilde{B}) > 2d(B_{m+1})$ for some m. Then $\tilde{B}$ must intersect one of the sets $B_{k+1}, \ldots, B_m$, because otherwise

$$d_m \ge d(\tilde{B}) > 2d(B_{m+1}) > d_m.$$

Let B_j be the first one of the balls $B_{k+1}, \dots, B_m$, which $\tilde{B}$ intersects. Then

$$\tilde{B} \cap B_j \neq \emptyset, \quad d(B_j) > d_{j-1}/2 \geq d(\tilde{B})/2 \quad \text{and} \quad k+1 \leq j \leq m.$$

Thus $\tilde{B} \subset 5B_j$ and (2.39) holds.



Finally let $\delta > 0$. When k is large enough, then $d(5B_j) \leq \delta$ for all $j \geq k$. Thus

$$\begin{aligned}
\mathcal{H}^s_\delta(E \setminus \bigcup_{i=1}^{\infty} B_i) &\leq \mathcal{H}^s_\delta(E \setminus \bigcup_{i=1}^{k} B_i) \\
&\leq \mathcal{H}^s_\delta(\bigcup_{j=k+1}^{\infty} 5B_j) \\
&\leq \omega_s 2^{-s} \sum_{j=k+1}^{\infty} d(5B_j)^s \\
&= \omega_s (5/2)^s \sum_{j=k+1}^{\infty} d(B_j)^s \to 0
\end{aligned}$$

as $k \to \infty$. Thus

$$\mathcal{H}^s_\delta(E \setminus \bigcup_{i=1}^{\infty} B_i) = 0$$

for all $\delta > 0$ and (2.37) holds. □

The next theorem is useful when studying local properties of s-sets.

Theorem 2.40. *Let* $0 < s < \infty$, $A \subset \mathbb{R}^n$, *and* $\mathcal{H}^s(A) < \infty$.

(a) $2^{-s} \leq \Theta^{*s}(A, a) \leq 1$ *for* $\mathcal{H}^s$*-a.e.* $a \in A$.

(b) If A *is* $\mathcal{H}^s$*-measurable, then* $\Theta^{*s}(A, a) = 0$ $\mathcal{H}^s$*-a.e.* $a \in \mathbb{R}^n \setminus A$ *.*

Remark 2.41.

1. The lower density $\Theta^s_*(A, a)$ could be zero for every $a \in \mathbb{R}^n$ even if $\mathcal{H}^s(A) > 0$; see [Ma, Exerc. 2, p. 99 and 4.12].
2. The upper bound 1 in (a) is sharp for all $s > 0$. The lower bound 2^{-s} is sharp for $0 < s \leq 1$, but it is not known whether it is sharp for $s > 1$; see [Ma, 6.4 (2)].

Proof. We first prove the left inequality of part (a). Observe first that

$$\{x \in A \colon \Theta^{*s}(A, x) < 2^{-s}\} = \bigcup_{k=1}^{\infty} \underbrace{\{x \in A \colon \mathcal{H}^s(A \cap \bar{B}(x, r)) < 2^{-s}\omega_s(1 - 1/k)r^s \quad \forall\, 0 < r < 1/k\}}_{=C_k}$$

and then show that $\mathcal{H}^s(C_k) = 0$ for all $k \in \mathbb{N}$. Fix $k \in \mathbb{N}$, $\varepsilon > 0$, and denote $C = C_k$. Cover C with the sets E_j, $j \in \mathbb{N}$, s.t. $0 < d(E_j) < 1/k$, $C \cap E_j \neq \emptyset$, and

$$2^{-s}\omega_s \sum_{j=1}^{\infty} d(E_j)^s \leq \mathcal{H}^s(C) + \varepsilon.$$

For every j choose $x_j \in C \cap E_j$ and denote $r_j = d(E_j)$. Because $C \cap E_j \subset A \cap \bar{B}(x_j, r_j)$, then

$$\begin{aligned}
\mathcal{H}^s(C) &\leq \sum_{j=1}^{\infty} \mathcal{H}^s(C \cap E_j) \\
&\leq \sum_{j=1}^{\infty} \mathcal{H}^s(A \cap \bar{B}(x_j, r_j)) \\
&\leq 2^{-s}\omega_s \sum_{j=1}^{\infty} (1 - 1/k) r_j^s \\
&= 2^{-s}\omega_s (1 - 1/k) \sum_{j=1}^{\infty} d(E_j)^s \\
&\leq (1 - 1/k)(\mathcal{H}^s(C) + \varepsilon).
\end{aligned}$$

Letting $\varepsilon \to 0$ we see that $\mathcal{H}^s(C) = 0$, because $1 - 1/k < 1$ and $\mathcal{H}^s(C) < \infty$.

We next prove the right hand side inequality of part (a). Because $\mathcal{H}^s$ is a Borel regular (outer measure), we may suppose that A is a Borel set. Then Corollary 1.44 shows that $\mathcal{H}^s \llcorner A$ is a Radon measure.

For $t > 1$, let

$$E = \{x \in A \colon \Theta^{*s}(A, x) > t\}.$$

We wish to show that $\mathcal{H}^s(E) = 0$. Let $\delta > 0$ and $\varepsilon > 0$. Because $\mathcal{H}^s \llcorner A$ is a Radon measure, there exists an open set $U \subset \mathbb{R}^n$ s.t. $E \subset U$ and

$$\mathcal{H}^s(A \cap U) < \mathcal{H}^s(E) + \varepsilon.$$

For every $x \in E$ there exists a radius $r_x < \delta/2$, for which $B(x, r_x) \subset U$ and a sequence of radii $r_i < r_x$, $r_i \to 0$, s.t.

$$\mathcal{H}^s(A \cap \bar{B}(x, r_i)) > t\omega_s r_i^s \quad \forall i \in \mathbb{N}. \tag{2.42}$$

(Note that the sequence r_i depends on x hence $r_i = r_i(x)$.) Next we apply Vitali's covering theorem to the Vitali covering $\mathcal{V} = \{\bar{B}(x, r_i) \colon x \in E,\ i \in \mathbb{N}\}$ of E. Therefore there exist disjoint closed balls $\{B_j\} \subset \mathcal{V}$ s.t.

$$\mathcal{H}^s(E \setminus \cup_j B_j) = 0. \tag{2.43}$$

Note that by (2.42)

$$t\omega_s 2^{-s} \sum_j d(B_j)^s \leq \sum_j \mathcal{H}^s(A \cap B_j) \leq \mathcal{H}^s(A \cap U) \leq \mathcal{H}^s(E) + \varepsilon < \infty,$$

and therefore the option (2.36) does not hold and, consequently, (2.43) follows. Hence

$$\mathcal{H}^s(E) + \varepsilon \geq t 2^{-s}\omega_s \sum_j d(B_j)^s \geq t\,\mathcal{H}^s_\delta(E \cap \cup_j B_j) \geq t\,\mathcal{H}^s_\delta(E),$$

where the last inequality follows from (2.43) and the subadditivity of $\mathcal{H}^s_\delta$. Letting $\varepsilon \to 0$ and $\delta \to 0$, we obtain $t\,\mathcal{H}^s(E) \le \mathcal{H}^s(E) < \infty$. Therefore $\mathcal{H}^s(E) = 0$, because $t > 1$.

Finally we prove (b): Let $t > 0$ and

$$B = \{x \in \mathbb{R}^n \setminus A \colon \Theta^{*s}(A, x) > t\}.$$

We prove that $\mathcal{H}^s(B) = 0$. Fix $\delta > 0$ and $\varepsilon > 0$. We apply part (b) of Theorem 1.24 to the Borel regular outer measure $\mathcal{H}^s \llcorner A$ (see Lemma 1.23). Because $(\mathcal{H}^s \llcorner A)(B) = 0$, then by 1.24 there exists an open $U \subset \mathbb{R}^n$ s.t. $B \subset U$ and $\mathcal{H}^s(A \cap U) < \varepsilon$. For every $x \in B$ there exists a radius $0 < r(x) < \delta$ s.t. $\bar{B}(x, r(x)) \subset U$ and

$$\mathcal{H}^s\big(A \cap \bar{B}(x, r(x))\big) > t\omega_s\big(r(x)\big)^s.$$

From the basic covering theorem 2.33 it follows that there exist (countably many) disjoint balls $B_i = \bar{B}(x_i, r(x_i))$ s.t. $B \subset \cup_i 5B_i$. Thus

$$\begin{aligned} t\,\mathcal{H}^s_{10\delta}(B) &\le t2^{-s}\omega_s \sum_i d(5B_i)^s \\ &= 5^s t 2^{-s}\omega_s \sum_i d(B_i)^s \\ &\le 5^s \sum_i \mathcal{H}^s(A \cap B_i) \\ &\le 5^s \mathcal{H}^s(A \cap U) \\ &\le 5^s \varepsilon. \end{aligned}$$

Letting $\varepsilon \to 0$ we see that $\mathcal{H}^s_{10\delta}(B) = 0$, which implies further that $\mathcal{H}^s(B) = 0$ as $\delta \to 0$. □

Corollary 2.44. *Let $A, B \subset \mathbb{R}^n$ be $\mathcal{H}^s$-measurable s.t. $B \subset A$ and $\mathcal{H}^s(A) < \infty$. Then for $\mathcal{H}^s$-a.e. $x \in B$ there holds:*

$$\Theta^{*s}(A, x) = \Theta^{*s}(B, x) \quad \textit{and} \quad \Theta^s_*(A, x) = \Theta^s_*(B, x).$$

Proof.

$$\frac{\mathcal{H}^s\big(A \cap \bar{B}(x, r)\big)}{\omega_s r^s} = \underbrace{\frac{\mathcal{H}^s\big((A \setminus B) \cap \bar{B}(x, r)\big)}{\omega_s r^s}}_{\xrightarrow{r \to 0+} 0 \ \mathcal{H}^s\text{-a.e. } x \in B} + \frac{\mathcal{H}^s\big(B \cap \bar{B}(x, r)\big)}{\omega_s r^s}.$$

□

Definition 2.45. A set $E \subset \mathbb{R}^n$ is *m-rectifiable*, $m \in \mathbb{N}$, if $\mathcal{H}^m(E) < \infty$ and there exists Lipschitz maps $f_i \colon \mathbb{R}^m \to \mathbb{R}^n$, $i \in \mathbb{N}$, such that

$$\mathcal{H}^m\left(E \setminus \bigcup_i f_i \mathbb{R}^m\right) = 0.$$

Usually the finiteness of $\mathcal{H}^s(E)$ is not required, in which case E is called *countably m-rectifiable*.

By the McShane-Whitney extension theorem 2.2 it is equivalent to say that

$$\mathcal{H}^m\left(E \setminus \bigcup_i f_i A_i\right) = 0,$$

where $A_i \subset \mathbb{R}^m$ and $f_i \colon A_i \to \mathbb{R}^n$ are Lipschitz. More importantly, by applying Rademacher's theorem and (a consequence of) Whitney's extension theorem, we have

Lemma 2.46. *Let $E \subset \mathbb{R}^n$ be a $\mathcal{H}^m$-measurable, with $\mathcal{H}^m(E) < \infty$. Then E is m-rectifiable if and only if there exist m-dimensional C^1-smooth submanifolds $M_i \subset \mathbb{R}^n$, $i \in \mathbb{N}$, such that*

$$\mathcal{H}^m\left(E \setminus \bigcup_i M_i\right) = 0.$$

Definition 2.47. A set $P \subset \mathbb{R}^n$ is *purely m-unrectifiable* if

$$\mathcal{H}^m(P \cap R) = 0$$

for all m-rectifiable $R \subset \mathbb{R}^n$.

Remark 2.48. The set E in Example 2.31 is 1-rectifiable whereas the set J in Example 2.30 is purely 1-unrectifiable.

Theorem 2.49. *Let $E \subset \mathbb{R}^n$ be $\mathcal{H}^m$-measurable, with $\mathcal{H}^m(E) < \infty$ (and $m \in \mathbb{N}$). Then there exist $\mathcal{H}^m$-measurable sets P and R such that R is m-rectifiable, P is purely m-unrectifiable,*

$$E = R \cup P \quad \textit{and} \quad R \cap P = \emptyset.$$

Proof. Set $M = \sup\{\mathcal{H}^m(R)\colon R \subset E,\ R \text{ is } m\text{-rectifiable}\}$. For each $i \in \mathbb{N}$, choose an m-rectifiable set R_i such that

$$\mathcal{H}^m(R_i) > M - 1/i.$$

Then we can choose $R = \cup_i R_i$ and $P = E \setminus R$. □

For $m, n \in \mathbb{N}$, with $m < n$, let $G(n,m)$ denote the (Grassmannian) space of all m-dimensional (vector) subspaces of $\mathbb{R}^n$.

Definition 2.50. We say that $V \in G(n,m)$ is an *approximate tangent space* of $E \subset \mathbb{R}^n$ at $a \in \mathbb{R}^n$ if

$$\Theta^{*m}(E,a) > 0$$

and for all $\delta > 0$

$$\lim_{r\to 0} \frac{1}{r^m}\mathcal{H}^m\left(\{x \in E \cap \bar{B}(a,r)\colon \operatorname{dist}(x-a,V) > \delta|x-a|\}\right) = 0.$$

If such a space exists, we denote it by T_aE or T_a^mE.

Remark 2.51. (a) For $V \in G(n,m)$, $a \in \mathbb{R}^n$, and $\delta > 0$ let

$$V_{a,\delta}^C = \{x \in \mathbb{R}^n\colon \operatorname{dist}(x-a,V) > \delta|x-a|\}.$$

We then notice that $V = T_a^mE$ if and only if $\Theta^{*m}(E,a) > 0$ and $\Theta^{*m}(E \cap V_{a,\delta}^C, a) = 0$ for all $1 > \delta > 0$. Note that $V_{a,\delta}^C = \emptyset$ for $\delta \geq 1$ since $\operatorname{dist}(x-a,V) \leq |x-a|$.

(b) If $m = 1$, the approximate tangent *line* T_a^1E is unique if exists, but for $m \geq 2$ T_a^mE need not be unique. However, for $\mathcal{H}^m$-measurable sets E, with $\mathcal{H}^m(E) < \infty$ and $m \geq 2$, the approximate tangent space T_a^mE is unique at $\mathcal{H}^m$-a.e. point $a \in E$ where such a space exists.

(c) The definition above differs from (and is weaker than) that in [LY, 3.3.3] or [Si, 11.2] where $V \in G(n,m)$ is said to be the approximate tangent space of an $\mathcal{H}^m$-measurable subset $E \subset \mathbb{R}^n$ (with $\mathcal{H}^m(E \cap K) < \infty$ for every compact $K \subset \mathbb{R}^n$) at $a \in \mathbb{R}^n$ if

$$\lim_{\lambda \to 0+} \int_{\eta_{a,\lambda}(E)} f(y) d\mathcal{H}^m(y) = \int_V f(y) d\mathcal{H}^m(y) \quad \forall f \in C_0(\mathbb{R}^n),$$

where $\eta_{a,\lambda} \colon \mathbb{R}^n \to \mathbb{R}^n$ is defined as $\eta_{a,\lambda}(y) = (y-a)/\lambda$ for $a, y \in \mathbb{R}^n,\ \lambda > 0$.

From Corollary 2.44 (see also Remark 2.51 (a)) we get:

Theorem 2.52. *Let $A \subset B \subset \mathbb{R}^n$ be $\mathcal{H}^m$-measurable with $\mathcal{H}^m(B) < \infty$. Then for $\mathcal{H}^m$-a.e. $x \in A$, $T_x^m A$ exists if and only if $T_x^m B$ exists. Furthermore, if exist, they are equal $\mathcal{H}^m$-a.e.*

In particular, if E is m-rectifiable and M_i's are m-dimensional C^1-submanifolds as in Lemma 2.46, then at $\mathcal{H}^m$-a.e. $x \in E \cap M_i$ the approximate tangent space of E is the same as the usual tangent space of M_i.

The following theorem characterizes rectifiable sets in terms of approximate tangent spaces; see [Ma, Chapter 15]. (This might be discussed in the home work classes.)

Theorem 2.53. *Let $E \subset \mathbb{R}^n$ be $\mathcal{H}^m$-measurable with $\mathcal{H}^m(E) < \infty$. Then E is m-rectifiable if and only if E has the approximate tangent space $T_a E \in G(n,m)$ for $\mathcal{H}^m$-a.e. $a \in E$.*

As a corollary, we have a characterization of purely unrectifiability.

Lemma 2.54. *Let $E \subset \mathbb{R}^n$ be $\mathcal{H}^m$-measurable with $\mathcal{H}^m(E) < \infty$. Then E is purely m-unrectifiable if and only if the set of those points $a \in E$ for which $T_a^m E$ exists is of $\mathcal{H}^m$-measure zero.*

Another deep characterization of purely unrectifiable sets is the following Besicovitch-Federer structure theorem.

Theorem 2.55. *Let Q be a countable union of sets with finite $\mathcal{M}^m$-measure. Then Q is purely m-unrectifiable if and only if $\mathcal{H}^m(P_V Q) = 0$ for almost all $V \in G(n,m)$. Here $P_V \colon \mathbb{R}^n \to V$ is the orthogonal projection and "almost all" refers to a natural probability Radon measure $\gamma_{n,m}$ on $G(n,m)$.*

For the proof; see e.g. [Ma, Theorem 18.1]. *Remark:* There is a natural probability Radon measure $\gamma_{n,m}$ on $G(n,m)$ that can be obtained from the general theory of Haar measures. Indeed, the group $O(n)$ of orthogonal linear maps $\mathbb{R}^n \to \mathbb{R}^n$ is compact and hence there exists a unique invariant Radon measure (Haar measure) θ_n such that $\theta_n(O(n)) = 1$ and

$$\theta_n(A) = \theta_n\big(\{gh \colon h \in A\}\big) = \theta_n\big(\{hg \colon h \in A\}\big)$$

for all $A \subset O(n)$ and $g \in O(n)$. The measure $\gamma_{n,m}$ is then obtained by fixing $V \in G(n,m)$ and setting

$$\gamma_{n,m}(A) = \theta_n\big(\{g \colon gV \in A\}\big), \quad A \subset G(n,m).$$

Being uniformly distributed $\gamma_{n,m}$ is independent of the choice of V.

Suppose that $E \subset \mathbb{R}^n$ is (countably) m-rectifiable. Theorem 2.53 enables us to define the gradient $\nabla^E f$ of a Lipschitz function $f \colon \mathbb{R}^n \to \mathbb{R}$ at $\mathcal{H}^m$-a.e. $x \in E$ as

$$\nabla^E f(x) = \sum_{i=1}^m \partial_{v_i} f(x) v_i, \tag{2.56}$$

where $(v_1, \dots, v_m)$ is an orthonormal basis of $T_x^m E$ and $\partial_{v_i} f(x)$ denotes the directional derivative of f in the direction v_i. Note that we can write

$$E = E_0 \sqcup \bigsqcup_{j=1}^{\infty} E_j,$$

where $\mathcal{H}^m(E_0) = 0$ and $E_j \subset M_j$, with M_j an m-dimensional C^1-submanifold of $\mathbb{R}^n$. Then $\nabla^E f(x) = \nabla^{M_j} f(x)$ whenever $x \in E_j$ and $f|M_j$ is differentiable at x (which holds $\mathcal{H}^m$-a.e. in M_j by Rademacher's theorem).

Having defined $\nabla^E f(x)$, we can define the linear map $d^E f_x \colon T_x^m E \to \mathbb{R}$ by

$$d^E f_x(v) = \langle v, \nabla^E f(x) \rangle, \quad v \in T_x^m E,$$

at all points where $T_x^m E$ and $\nabla^E f(x)$ exist. Above $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{R}^n$.

If $f = (f_1, \dots, f_N) \colon \mathbb{R}^n \to \mathbb{R}^N$ is Lipschitz, we define a linear map $d^E f_x \colon T_x^m E \to \mathbb{R}^N$ by

$$d^E f_x(v) = \sum_{j=1}^{N} \langle v, \nabla^E f_j(x) \rangle e_j,$$

where $e_1, \dots, e_N$ is the standard basis of $\mathbb{R}^N$. If $N \geq m$, we define the Jacobian of f, denoted by $J_f^E(x)$, for $\mathcal{H}^m$-a.e. $x \in E$ by

$$J_f^E(x) = \sqrt{\det (d^E f_x)^* \circ d^E f_x}. \tag{2.57}$$

Then we have the general area formula

$$\int_A J_f^E \, d\mathcal{H}^m = \int_{\mathbb{R}^N} \mathcal{H}^0\big(A \cap f^{-1}(y)\big) d\mathcal{H}^m(y) \tag{2.58}$$

for every $\mathcal{H}^m$-measurable $A \subset E$. Similarly, in the case $N < m$, we can define

$$J_f^E(x) = \sqrt{\det (d^E f_x) \circ (d^E f_x)^*}$$

and obtain the general co-area formula

$$\int_A J_f^E(x) d\mathcal{H}^m(x) = \int_{\mathbb{R}^N} \mathcal{H}^{m-N}\big(A \cap f^{-1}(y)\big) d\mathcal{H}^N(y)$$

for every $\mathcal{H}^m$-measurable set. The following theorem will be useful in studying "slices" of currents.

Theorem 2.59. *Let $E \subset \mathbb{R}^n$ be m-rectifiable and $f \colon \mathbb{R}^n \to \mathbb{R}$ Lipschitz. Then for m_1-a.e. $t \in \mathbb{R}$,*

(1) $E_t := f^{-1}(t) \cap E$ is $(m-1)$-rectifiable and

(2) for $\mathcal{H}^{m-1}$-a.e. $x \in E_t$, tangent spaces $T_x^{m-1} E_t$ and $T_x^m E$ exist, $T_x^{m-1} \subset T_x^m E$, and

$$T_x^m E = \{ y + \lambda \nabla^E f(x) \colon y \in T_x^{m-1} E_t, \ \lambda \in \mathbb{R} \}.$$

(3) For every nonnegative $\mathcal{H}^m$-measurable $g \colon E \to \mathbb{R}$, we have (the co-area formula)

$$\int_{-\infty}^{\infty} \int_{E_t} g \, d\mathcal{H}^{m-1} \, dt = \int_E |\nabla^E f| g \, d\mathcal{H}^m.$$

For the proof; see [Si, p. 68-69 and 28.1]. Here we just sketch the proof:
The finiteness of $\mathcal{H}^{m-1} E_t$ for a.e. $t \in \mathbb{R}$ follows from Lemma 2.25 (iv). We can write

$$E = E_0 \sqcup \bigsqcup_{j=1}^{\infty} E_j,$$

where $\mathcal{H}^m(E_0) = 0$ and $E_j \subset M_j$, with M_j an m-dimensional C^1-submanifold of $\mathbb{R}^n$. Then $\mathcal{H}^{m-1}\big(E_0 \cap f^{-1}(t)\big) = 0$ for a.e. $t \in \mathbb{R}$. Hence it is enough to prove the claims for $E = M$, where M is an m-dimensional C^1-submanifold of $\mathbb{R}^n$, with $\mathcal{H}^m(M) < \infty$. Applying the implicit function theorem (and using local coordinates), we may assume that $M \subset \mathbb{R}^m$, with $m_m(M) < \infty$. Rademacher's theorem and Whitney extension theorem imply that, for every $\varepsilon > 0$, there exists $g_\varepsilon \in C^1$ such that

$$m_m\left(\{x \in M \colon f(x) \neq g_\varepsilon(x) \text{ or } \nabla f(x) \neq \nabla g_\varepsilon(x)\}\right) < \varepsilon.$$

Applying this with $\varepsilon = 1/i,\ i \in \mathbb{N}$, the problems are reduced to the case $f \in C^1$. Sard's theorem implies that

$$m_1\left(\{f(x) \colon |\nabla f(x)| = 0\}\right) = 0.$$

Thus we may assume that $\nabla f(x) \neq 0$ for every $x \in M$. Now the implicit function theorem implies that the level sets $M_t = \{x \in M \colon f(x) = t\}$ are locally $(m-1)$-dimensional C^1 submanifolds, hence $(m-1)$-rectifiable. This proves (1). The claim (2) follows from the facts that $\nabla^M f(x) \in T_x^m M$ and $\nabla^M f(x) \perp T_x^{m-1} M_t$. Finally, (3) is a generalization of the co-area formula.

3 Varifolds

From Wikipedia: Varifolds were first introduced by L.C. Young in 1951, under the name "generalized surfaces". Frederick Almgren slightly modified the definition in his mimeographed notes (Almgren 1965) and coined the name varifold: he wanted to emphasize that these objects are substitutes for ordinary manifolds in problems of the calculus of variations. The modern approach to the theory was based on Almgren's notes and laid down by William Allard (Allard 1972).

Varifolds can be interpreted as measure-theoretic generalizations of smooth manifolds and they generalize the idea of rectifiable currents.

3.1 Basic definitions

We start with introducing a metric (and hence a topology) on the Grassmannian space

$$G(n,m) = \{V \subset \mathbb{R}^n \colon V \ m\text{-dimensional subspace of } \mathbb{R}^n\}.$$

For $V, W \in G(n,m)$, define

$$d(V,W) = \|P_V - P_W\| = \sup\{|P_V x - P_W x| \colon x \in \mathbb{R}^n, |x| = 1\},$$

where $P_V \colon \mathbb{R}^n \to V$ is the orthogonal projection onto V. With this metric $G(n,m)$ is a compact metric space.

Definition 3.2. Let $U \subset \mathbb{R}^n$ be open and $0 \leq m \leq n$ integers. A Radon (outer) measure on $U \times G(n,m)$ is called an *m-dimensional varifold* (or m-varifold) in U. The set of m-dimensional varifolds in U is denoted by $V_m(U)$.

Hence

$$\begin{aligned} V_m(U) &= \{V\colon V \text{ a Radon outer measure on } U \times G(n,m)\} \\ &= \{\mu\colon \mu \text{ Borel regular outer measure on } U \times G(n,m), \\ &\qquad \mu(K \times G(n,m)) < \infty \ \forall \text{ compact } K \subset U\}. \end{aligned}$$

We equip $V_m(U)$ with the weak topology (the following is just the rephrase of Definition 1.65 for Radon outer measures):

Definition 3.3. The sequence $V_i \in V_m(U)$ is said to converge to $V \in V_m(U)$ (as varifolds), denoted by $V_i \to V$, if $V_i \rightharpoonup V$ as Radon (outer) measures, i.e.

$$\int_{U \times G(n,m)} f \, dV_i \to \int_{U \times G(n,m)} f \, dV \quad \forall f \in C_0(U \times G(n,m)).$$

Definition 3.4. For each $V \in V_m(U)$ we define the measure $\|V\|$ and its m-dimensional density $d(V,\cdot)$ in U by setting

$$\begin{aligned} \|V\|(A) &= V\big(A \times G(n,m)\big) \quad \text{for Borel sets } A \subset U, \\ d(V,a,r) &= \frac{\|V\|\big(\bar{B}(a,r)\big)}{\omega_m r^m}, \ r > 0, \\ d(V,a) &= \lim_{r \to 0} d(V,a,r) \quad \text{for } a \in U \text{ if the limit exists.} \end{aligned}$$

The measure $\|V\|$ is also called the *weight (measure)* of V and denoted by μ_V. The *mass* of V is defined as $\mathbf{M}_V = \|V\|(U)$.

We abbreviate

$$G_{n,m}(U) = U \times G(n,m), \ G_{n,m} = G_{n,m}(\mathbb{R}^n).$$

Example 3.5. Let $E \subset \mathbb{R}^n$ be an $\mathcal{H}^m$-measurable m-rectifiable set. Then E has the approximate tangent space $T_x^m E \in G(n,m)$ for $\mathcal{H}^m$-a.e. $x \in E$. Define

$$V_E(A) = \mathcal{H}^m\big(\{x \in E\colon (x, T_x^m E) \in A\}\big)$$

for $A \subset G_{n,m}$. Then V_E is an m-varifold, $\|V_E\| = \mathcal{H}^m \llcorner E$, and $\mathbf{M}_{V_E} = \mathcal{H}^m(E)$. Moreover,

$$\int_{G_{n,m}} f \, dV_E = \int_E f(x, T_x^m E) \, d\mathcal{H}^m(x)$$

for all $f \in C_0(G_{n,m})$.

Definition 3.6. Let E and $\tilde{E}$ be $\mathcal{H}^m$-measurable and (countably) m-rectifiable subsets of $\mathbb{R}^n$, and let θ (resp. $\tilde{\theta}$) be nonnegative and locally $\mathcal{H}^m$-integrable in E (resp. $\tilde{E}$). We say that (E,θ) and $(\tilde{E},\tilde{\theta})$ are equivalent if

$$\mathcal{H}^m\big((E \setminus \tilde{E}) \cup (\tilde{E} \setminus E)\big) = 0$$

and $\theta = \tilde{\theta}$ $\mathcal{H}^m$-a.e. in $E \cap \tilde{E}$. A *(countably) rectifiable m-varifold* $V_{E,\theta} = V(E,\theta)$ is the equivalence class of a pair (E,θ) as above and (E,θ) is called a *representative* for V. If θ is integer valued, $V(E,\theta)$ is called an integer multiplicity rectifiable m-varifold, or briefly an *integer m-varifold*.

We adopt the convention that $\theta \equiv 0$ in $\mathbb{R}^n \setminus E$. Associated to a rectifiable m-varifold $V = V(E, \theta)$ there is a Radon measure μ_V, called the *weight measure* of V, defined by

$$\mu_V = \mathcal{H}^m \llcorner \theta, \tag{3.7}$$

that is

$$\mu_V(A) = \int_{A \cap E} \theta \, d\mathcal{H}^m$$

for $\mathcal{H}^m$-measurable sets A. The *mass* of $V = V(E, \theta)$ is

$$\mathbf{M}_V = \mu_V(\mathbb{R}^n) = \int_{E \cap \mathbb{R}^n} \theta \, d\mathcal{H}^m = (\mathcal{H}^m \llcorner \theta)(\mathbb{R}^n).$$

Every countably rectifiable m-varifold $V(E, \theta)$ induces an m-varifold $V_{E,\theta}$ by

$$V_{E,\theta}(A) = \int_{\{x \in E \colon (x, T_x^m E) \in A\}} \theta \, d\mathcal{H}^m(x), \quad A \in G_{n,m}.$$

3.8 First and second variation formulae

Next we will study how the mass $\mathbf{M}_V$ of an m-varifold $V \in V_m(U)$ (resp. of a rectifiable m-varifold $V = (E, \theta)$) behaves under a perturbation by a 1-parameter family of diffeomorphisms. To get an idea, let us consider first (a less abstract setting of) an m-dimensional C^1-smooth submanifold $M \subset \mathbb{R}^n$.

Remark 3.9. Let M be an m-dimensional C^1-smooth submanifold of $\mathbb{R}^n$. For every point $x \in M$ there exist an open neighborhood $A \subset \mathbb{R}^n$ of x and a C^1-diffeomorphism $\varphi \colon A \to A'$ onto an open set $A' \subset \mathbb{R}^n$ such that $\varphi(A \cap M)$ is an open subset of $\mathbb{R}^m \times \{0\} \subset \mathbb{R}^m \times \mathbb{R}^{n-m}$. Note that $T_x M = (d\varphi_x)^{-1} \mathbb{R}^m$.

Let then $U \subset \mathbb{R}^n$ be open such that $U \cap M \neq \emptyset$ and $\mathcal{H}^m(C \cap M) < \infty$ for every compact $C \subset U$. Let $\{\phi_t\}$, $-1 < t < 1$, be a 1-parameter family of diffeomorphisms $\phi_t \colon U \to U$ such that

$$\begin{aligned} &\phi \colon (-1,1) \times U \to U, \ \phi(t,x) = \phi_t(x), \text{ is } C^2, \\ &\phi_0(x) = x \ \forall x \in U, \text{ and} \\ &\phi_t(x) = x \ \forall x \in U \setminus K \text{ and } t \in (-1,1), \end{aligned} \tag{3.10}$$

for some compact $K \subset U$. Define mappings $X = (X^1, \ldots, X^n) \colon U \to \mathbb{R}^n$ and $Z = (Z^1, \ldots, Z^n) \colon U \to \mathbb{R}^n$ by

$$X(x) = \frac{\partial \phi(t,x)}{\partial t}_{|t=0} \quad \text{and} \quad Z(x) = \frac{\partial^2 \phi(t,x)}{\partial t^2}_{|t=0}. \tag{3.11}$$

Then

$$\phi_t(x) = x + tX(x) + \frac{t^2}{2} Z(x) + O(t^3), \tag{3.12}$$

where $O(t^3) \in \mathbb{R}^n$, with $|O(t^3)| \le c|t|^3$. Since $\phi_t(x) = x$ for $x \in U \setminus K$, the maps X and Z are compactly supported.

Definition 3.13. Let $M_t = \phi_t(M \cap K)$. The *first* and *second variations* of M (with respect to a 1-parameter family $\{\phi_t\}$) are defined as

$$\frac{d}{dt}\mathcal{H}^m(M_t)_{|t=0} \quad \text{and} \quad \frac{d^2}{dt^2}\mathcal{H}^m(M_t)_{|t=0},$$

respectively.

By the area formula

$$\mathcal{H}^m(M_t) = \mathcal{H}^m\big(\phi_t(M \cap K)\big) = \int_{M\cap K} J_{\psi_t}\, d\mathcal{H}^m,$$

where $\psi_t = \phi_t | M \cap U$. Since we can change the order of integration and differentiation, the computation of the first and second variations reduces to calculating

$$\frac{\partial}{\partial t} J_{\psi_t|t=0} \quad \text{and} \quad \frac{\partial^2}{\partial t^2} J_{\psi_t|t=0}.$$

For that purpose, let us fix orthonormal bases $\tau_1, \dots, \tau_m$ of T_xM for $x \in M$ and $e_1, \dots, e_n$ of $\mathbb{R}^n$. We define the (induced) linear map $d\psi_{t,x} \colon T_xM \to \mathbb{R}^n$ of ψ_t at $x \in M$ by

$$d\psi_{t,x}(\tau) = \partial_\tau \phi_t(x) = \partial_\tau \psi_t(x), \quad \tau \in T_xM.$$

By (3.12), we have

$$d\psi_{t,x}(\tau) = \tau + t\partial_\tau X(x) + \frac{t^2}{2}\partial_\tau Z(x) + O(t^3).$$

Writing the basis vectors $\tau_j,\ j = 1, \dots, m$, as

$$\tau_j = \sum_{i=1}^n \tau_j^i e_i,$$

we can express the matrix $(a_{ij})_{n\times m}$ of $d\psi_{t,x}$ w.r.t. bases $\tau_1, \dots, \tau_m$ of $T_xM,\ x \in M$, and $e_1, \dots, e_n$ of $\mathbb{R}^n$ as

$$a_{ij} = \tau_j^i + t\partial_{\tau_j} X^i + \frac{t^2}{2}\partial_{\tau_j} Z^i + O(t^3).$$

Consequently, the matrix of $(d\psi_{t,x})^* \circ (d\psi_{t,x})$ is $(b_{ij})_{m\times m}$, where

$$\begin{aligned} b_{ij} &= \sum_{k=1}^n a_{ki}a_{kj} \\ &= \delta_{ij} + t\left(\langle \tau_i, \partial_{\tau_j} X\rangle + \langle \tau_j, \partial_{\tau_i} X\rangle\right) + t^2\left(\frac{1}{2}\left(\langle \tau_i, \partial_{\tau_j} Z\rangle + \langle \tau_j, \partial_{\tau_i} Z\rangle\right) + \langle \partial_{\tau_i} X, \partial_{\tau_j} X\rangle\right) + O(t^3). \end{aligned}$$

Next we apply the formula

$$\begin{aligned} \det\left(I + tA + t^2B\right) &= \exp\log\det\left(I + tA + t^2B\right) \\ &= \exp\operatorname{Tr}\left(\log\left(I + tA + t^2B\right)\right) \\ &= \exp\operatorname{Tr}\left(tA + t^2B - \frac{1}{2}(tA + t^2B)^2 + O(t^3)\right) \\ &= \exp\left(t\operatorname{Tr}A + t^2\operatorname{Tr}B - \frac{1}{2}\operatorname{Tr}\left(t^2A^2 + 2t^3AB + t^4B^2\right) + O(t^3)\right) \\ &= \exp\left(t\operatorname{Tr}A + t^2\operatorname{Tr}B - \frac{1}{2}t^2\operatorname{Tr}A^2 + O(t^3)\right) \\ &= 1 + t\operatorname{Tr}A + t^2\operatorname{Tr}B - \frac{1}{2}t^2\operatorname{Tr}A^2 + \frac{1}{2}t^2(\operatorname{Tr}A)^2 + O(t^3) \end{aligned}$$

for symmetric square matrices $I = (\delta_{ij})$ = the identity matrix, $A = (A_{ij})$, and $B = (B_{ij})$, where

$$\begin{aligned}A_{ij} &= \langle \tau_i, \partial_{\tau_j} X\rangle + \langle \tau_j, \partial_{\tau_i} X\rangle = A_{ji} \quad \text{and} \\ B_{ij} &= \frac{1}{2}\left(\langle \tau_i, \partial_{\tau_j} Z\rangle + \langle \tau_j, \partial_{\tau_i} Z\rangle\right) + \langle \partial_{\tau_i} X, \partial_{\tau_j} X\rangle,\end{aligned}$$

to obtain

$$\begin{aligned}J^2_{\psi_t}(x) &= \det(d\psi_{t,x})^* \circ (d\psi_{t,x}) = \det(b_{ij}) \\ &= 1 + 2t\sum_{i=1}^m \langle \tau_i, \partial_{\tau_i} X\rangle + t^2 \sum_{i=1}^m \left(\langle \tau_i, \partial_{\tau_i} Z\rangle + |\partial_{\tau_i} X|^2\right) + 2t^2\left(\sum_{i=1}^m \langle \tau_i, \partial_{\tau_i} X\rangle\right)^2 \\ &\quad - \frac{1}{2}t^2 \sum_{i,j=1}^m \left(\langle \tau_i, \partial_{\tau_j} X\rangle + \langle \tau_j, \partial_{\tau_i} X\rangle\right)^2 + O(t^3) \\ &= 1 + 2t \operatorname{div}_M X + t^2 \operatorname{div}_M Z + t^2 \sum_{i=1}^m |\partial_{\tau_i} X|^2 + 2t^2 (\operatorname{div}_M X)^2 \\ &\quad - t^2 \sum_{i,j=1}^m \langle \tau_i, \partial_{\tau_j} X\rangle^2 - t^2 \sum_{i,j=1}^m \langle \tau_i, \partial_{\tau_j} X\rangle\langle \tau_j, \partial_{\tau_i} X\rangle + O(t^3) \\ &= 1 + 2t \operatorname{div}_M X + t^2\left(\operatorname{div}_M Z + 2(\operatorname{div}_M X)^2 + \sum_{i=1}^m \left|(\partial_{\tau_i} X)^\perp\right|^2 - \sum_{i,j=1}^m \langle \tau_i, \partial_{\tau_j} X\rangle\langle \tau_j, \partial_{\tau_i} X\rangle\right) \\ &\quad + O(t^3),\end{aligned}$$

where

$$(\partial_{\tau_i} X)^\perp = \partial_{\tau_i} X - \sum_{j=1}^m \langle \tau_j, \partial_{\tau_i} X\rangle \tau_j$$

is the normal component of $\partial_{\tau_i} X$ (normal to M). Above $\operatorname{div}_M X$ is the divergence of X (at $x \in M$) with respect to M defined as

$$\operatorname{div}_M X = \sum_{i=1}^m \langle \tau_i, \partial_{\tau_i} X\rangle.$$

Finally, using

$$\sqrt{1+s} = 1 + \frac{1}{2}s - \frac{1}{8}s^2 + O(s^3),$$

we get

$$\begin{aligned}J_{\psi_t}(x) &= 1 + t \operatorname{div}_M X + \frac{t^2}{2}\left(\operatorname{div}_M Z + 2(\operatorname{div}_M X)^2 + \sum_{i=1}^m \left|(\partial_{\tau_i} X)^\perp\right|^2 - \sum_{i,j=1}^m \langle \tau_i, \partial_{\tau_j} X\rangle\langle \tau_j, \partial_{\tau_i} X\rangle\right) \\ &\quad - \frac{t^2}{8}(2\operatorname{div}_M X)^2 + O(t^3) \\ &= 1 + t \operatorname{div}_M X + \frac{t^2}{2}\left(\operatorname{div}_M Z + (\operatorname{div}_M X)^2 + \sum_{i=1}^m \left|(\partial_{\tau_i} X)^\perp\right|^2 - \sum_{i,j=1}^m \langle \tau_i, \partial_{\tau_j} X\rangle\langle \tau_j, \partial_{\tau_i} X\rangle\right) \\ &\quad + O(t^3).\end{aligned}$$

Hence

$$\frac{\partial}{\partial t} J_{\psi_t|t=0} = \operatorname{div}_M X,$$

and therefore, by the area formula, we obtain the *first variation formula*

$$\begin{aligned} \frac{d}{dt}\mathcal{H}^m(M_t)_{|t=0} &= \int_{M\cap K} \frac{\partial}{\partial t} J_{\psi_t|t=0}\, d\mathcal{H}^m \\ &= \int_{M\cap K} \operatorname{div}_M X\, d\mathcal{H}^m \\ &= \int_M \operatorname{div}_M X\, d\mathcal{H}^m, \end{aligned} \tag{3.14}$$

where the last equality holds since $X \equiv 0$ in $M \setminus K$. Similarly, we get the *second variation formula*

$$\begin{aligned} &\frac{d^2}{dt^2}\mathcal{H}^m(M_t)_{|t=0} \\ &= \int_M \Big(\operatorname{div}_M Z + (\operatorname{div}_M X)^2 + \sum_{i=1}^m \big|\big(\partial_{\tau_i} X\big)^\perp\big|^2 - \sum_{i,j=1}^m \langle \tau_i, \partial_{\tau_j} X\rangle\langle \tau_j, \partial_{\tau_i} X\rangle\Big) d\mathcal{H}^m \end{aligned} \tag{3.15}$$

Definition 3.16. An m-dimensional C^1-smooth submanifold $M \subset \mathbb{R}^n$ is *stationary* in an open set $U \subset \mathbb{R}^n$ if $\mathcal{H}^m(M \cap C) < \infty$ for every compact $C \subset U$ and if

$$\frac{d}{dt}\mathcal{H}^m(M_t)_{|t=0} = 0$$

for $M_t = \phi_t(M \cap K)$ whenever ϕ_t and K are as in (3.10).

By the first variation formula (3.14), M is stationary in U if and only if

$$\int_M \operatorname{div}_M X\, d\mathcal{H}^m = 0$$

for every C^1-smooth $X\colon U \to \mathbb{R}^n$ with compact support in U. Indeed, every such X generates a 1-parameter family of C^2-diffeomorphisms $\{\phi_t\}$ satisfying (3.10), with $K = \operatorname{supp} X$, as the flow of X. More precisely, for every $x \in U$, $t \mapsto \phi_t(x)$ is the integral curve of X starting at x, that is $\phi_0(x) = x$ and

$$\frac{d}{dt}\phi_t(x) = X\big(\phi_t(x)\big).$$

Remark 3.17. If M is an m-dimensional C^2-smooth submanifold of $\mathbb{R}^n$, $m < n$, and $U \subset \mathbb{R}^n$ is open such that $\bar{U} \cap M$ is compact, then M is stationary in U if and only if $H \equiv 0$ in $M \cap U$, where H is the mean curvature vector of M. The mean curvature of M will be discussed in a home work session.

Next we will generalize the first variation formula for rectifiable m-varifolds. Let $V = V(E, \theta)$ be a rectifiable m-varifold in an open set $U \subset \mathbb{R}^n$. We suppose for simplicity that

$$\theta(x) \ge 1 \tag{3.18}$$

for $\mathcal{H}^m$-a.e. $x \in E$. This restriction is made to avoid discussions on approximate tangent spaces (and hence Jacobians) with respect to multiplicity θ. We conclude from Theorem 2.52 that $T_x^m E$ and $T_x^m \tilde{E}$ exists and are equal for $\mathcal{H}^m$-a.e. $x \in E \cap \tilde{E}$ if $(\tilde{E}, \tilde{\theta})$ is another representative for V.

Therefore we can define the approximate tangent space of V at x by setting $T_xV = T_x^mE$. Suppose then that $f\colon U \to U'$ is a Lipschitz mapping to an open set $U' \subset \mathbb{R}^N$, $N \geq n$, with the Jacobian J_f^E defined in (2.57). We notice that $J_f^E(x) = J_f^{\tilde{E}}(x)$ for $\mathcal{H}^m$-a.e. $x \in E \cap \tilde{E}$, and hence we may denote it by J_f^V. By the general area formula (2.58), we have

$$\int_A g\, J_f^E\, d\mathcal{H}^m = \int_{fE} \sum_{x \in A \cap f^{-1}(y)} g(x)\, d\mathcal{H}^m = \int_{fE} \left(\int_{A \cap f^{-1}(y)} g\mathcal{H}^0 \right) d\mathcal{H}^m \tag{3.19}$$

for every nonnegative $\mathcal{H}^m$-measurable g on E and $\mathcal{H}^m$-measurable $A \subset E$. Clearly fE is an m-rectifiable subset of U'. We assume, moreover, that $f\colon U \to U'$ is *proper*, that is $f^{-1}K \subset U$ is compact for every compact $K \subset U'$. Then we define θ' on U' by setting

$$\theta'(y) = \sum_{x \in E \cap f^{-1}} \theta(x) = \int_{E \cap f^{-1}(y)} \theta\, d\mathcal{H}^0$$

and the *image (or push-forward)*

$$f_\# V = V(fE, \theta').$$

Since

$$\int_K \theta'\, d\mathcal{H}^m = \int_{fE \cap K} \theta'\, d\mathcal{H}^m = \int_{E \cap f^{-1}K} \theta J_f^E\, d\mathcal{H}^m$$

for every compact $K \subset U'$, we see that θ' is locally $\mathcal{H}^m$-integrable in U'. Hence $f_\# V$ is a rectifiable m-varifold in U' with multiplicity θ'. Moreover,

$$\mathbf{M}_{f_\# V} = \int_{fE} \theta'\, d\mathcal{H}^m = \int_E J_f^E \theta\, d\mathcal{H}^m.$$

Now we are ready to define the first variation of V. Let $\{\phi_t\}$ be a 1-parameter family of diffeomorphisms $\phi_t\colon U \to U$ as in (3.10). We denote $V \llcorner K = V(E \cap K, \theta | K)$, where $K \subset U$ is the compact set in (3.10). Then

$$\mathbf{M}_{\phi_{t\#}(V \llcorner K)} = \int_{E \cap K} J_{\phi_t}^E \theta\, d\mathcal{H}^m$$

and we can compute the *first variation*

$$\frac{d}{dt} \mathbf{M}_{\phi_{t\#}(V \llcorner K) | t=0}$$

exactly as in the case of C^1-submanifolds and obtain

$$\frac{d}{dt} \mathbf{M}_{\phi_{t\#}(V \llcorner K) | t=0} = \int_E \operatorname{div}_E X\, d\mu_V, \tag{3.20}$$

where X is as in (3.11) and $\operatorname{div}_E X$ is the divergence of X with respect to E, defined as

$$\operatorname{div}_E X(x) = \sum_{i=1}^m \langle \tau_i, \partial_{\tau_i} X(x) \rangle,$$

with $\tau_1, \ldots, \tau_m$ an orthonormal bases of T_x^mE.

As in the case of C^1-submanifolds, we define

Definition 3.21. A rectifiable m-varifold $V = V(E,\theta)$ is *stationary* in an open set $U \subset \mathbb{R}^n$ if

$$\int_E \operatorname{div}_E X \, d\mu_V = 0$$

for any C^1-smooth $X\colon U \to \mathbb{R}^n$ with compact support in U.

We also generalize the notion of mean curvature as follows:

Definition 3.22. Let $V = V(E,\theta)$ be a rectifiable m-varifold in an open set $U \subset \mathbb{R}^n$. Suppose $H\colon E\cap U \to \mathbb{R}^n$ is locally μ_V-integrable. We say that $V = V(E,\theta)$ has *generalized mean curvature* H in U if

$$\int_U \operatorname{div}_E X \, d\mu_V = -\int_U \langle X, H\rangle \, d\mu_V$$

whenever $X\colon U \to \mathbb{R}^n$ is a C^1 with compact support in U.

Hence a rectifiable m-varifold $V = V(E,\theta)$ is stationary in an open set $U \subset \mathbb{R}^n$ if and only if it has zero generalized mean curvature in U.

Next we will introduce the variation of a (general) varifold. For that purpose, let $U, U' \subset \mathbb{R}^n$ be open, $V \in V_m(U)$ an m-varifold in U, and suppose that $f\colon U \to U'$ is a C^1-diffeomorphism. Recall that an m-varifold in an open set $U \subset \mathbb{R}^n$ is a Radon (outer) measure on $G_{n,m}(U) = U \times G(n,m)$. First we define the *push forward of* V *under* f by setting for Borel sets $B \subset G_{n,m}(U')$

$$f_\# V(B) = \int_{F^{-1}(B)} J_f(x,E) dV(x,E),$$

where $F\colon G_{n,m}(U) \to G_{n,m}(U')$ is defined by

$$F(x,E) = \big(f(x), df_x E\big)$$

and

$$J_f(x,E) = \Big(\det\big(df_x|E\big)^* \circ \big(df_x|E\big)\Big)^{1/2}, \quad (x,E) \in G_{n,m}(U).$$

Note that $df_x\colon \mathbb{R}^n \to \mathbb{R}^n$ is an invertible linear map for all $x \in U$ since $f\colon U \to U'$ is a C^1-diffeomorphism. In particular, $df_x|E\colon E \to df_x E \in G(n,m)$ is invertible. For a Borel set $A \subset U$, the restriction $V\llcorner G_{n,m}(A)$ is the Radon measure in $G_{n,m}(U)$ defined as

$$\big(V\llcorner G_{n,m}(A)\big)(B) = V\big(B \cap G_{n,m}(A)\big), \quad B \subset G_{n,m}(U).$$

Definition 3.23. Let V be an m-varifold in an open set $U \in \mathbb{R}^n$ and let $C_0^1(U,\mathbb{R}^n)$ be the space of C^1-mappings $X\colon U \to \mathbb{R}^n$ with compact support in U. Then the *first variation* of V is the linear functional $\delta V\colon C_0^1(U,\mathbb{R}^n) \to \mathbb{R}$,

$$\delta V(X) = \frac{d}{dt}\mathbf{M}_{\phi_{t\#}(V\llcorner G_{n,m}(K))|t=0},$$

where $\{\phi_t\}$ is a 1-parameter family of diffeomorphisms $U \to U$ associated to $X \in C_0^1(U,\mathbb{R}^n)$ as in (3.10) and (3.11), that is $\phi = \phi(\cdot,\cdot)$ is the flow of X.

Again, exactly the same computation as in smooth case gives

$$\delta V(X) = \int_{G_{n,m}(U)} \operatorname{div}_S X(x) \, dV(x,S), \tag{3.24}$$

where, for any $(x, S) \in G_{(n,m)}(U)$, $\operatorname{div}_S X$ is the divergence of X with respect to S, defined as

$$\operatorname{div}_S X(x) = \sum_{i=1}^{m} \langle \tau_i, \partial_{\tau_i} X(x) \rangle,$$

with $\tau_1, \ldots, \tau_m$ an orthonormal bases of S.

Definition 3.25. A varifold $V \in V_m(U)$ is said to be *stationary* if $\delta V(X) = 0$ for every $X \in C_0^1(U, \mathbb{R}^n)$.

More generally, V is said to have *locally bounded first variation* if for each $W \Subset U$ there exists a constant $c < \infty$ such that

$$|\delta V(X)| \le c \sup_U |X| \quad \forall X \in C_0^1(U, \mathbb{R}^n), \text{ with } \operatorname{supp} X \subset W.$$

Definition 3.26. For any $V \in V_m(U)$, we define the set function $\|\delta V\| \colon \mathcal{P}(U) \to [0, \infty]$ by

$$\|\delta V\|(U') = \sup\{\delta V(X) \colon X \in C_0^1(U, \mathbb{R}^n), |X| \le 1, \operatorname{supp} X \subset U'\}$$

for open sets $U' \subset U$, and then

$$\|\delta V\|(A) = \inf\{\|\delta V\|(U') \colon U' \subset U \text{ open}, \ A \subset U'\}$$

for $A \subset U$.

We note that $\|\delta V\|$ is a metric outer measure. If V has locally bounded first variation, then $\|\delta V\|$ is a Radon measure by Theorem 1.31 and, moreover, by the general Riesz representation theorem 1.34, there exists a $\|\delta V\|$-measurable mapping $\eta_V \colon U \to \mathbb{S}^{n-1}$ such that

$$\delta V(X) = -\int_U \langle X, \eta_V \rangle \, d\|\delta V\| \tag{3.27}$$

for all $X \in C_0^1(U, \mathbb{R}^n)$. [Use the Hahn-Banach theorem to extend $\delta V \colon C_0^1(U, \mathbb{R}^n)$ to a linear functional on $C_0(U, \mathbb{R}^n)$ and remember the construction of the Radon measure μ in the proof of the Riesz representation theorem to note that μ is, in fact, $\|\delta V\|$.]

Recall that the weight (measure) $\mu_V = \|V\|$ is defined as

$$\mu_V(A) = V\big(A \times G(n, m)\big)$$

for all Borel sets $A \subset U$. By the Radon-Nikodym theorem (see e.g. [Ma, 2.17], [Si, 4.7], or [Ho, 5.31]), the *Radon-Nikodym derivative*

$$D_{\mu_V} \|\delta V\|(x) = \lim_{r \to 0} \frac{\|\delta V\|\big(\bar{B}(x, r)\big)}{\mu_V\big(\bar{B}(x, r)\big)}$$

exists for μ_V-a.e. $x \in U$ and

$$\int_U \langle X, \eta_V \rangle \, d\|\delta V\| = \int_U \langle X, H_V \rangle \, d\mu_V + \int_U \langle X, \eta_V \rangle \, d\sigma,$$

where $\sigma = \|\delta V\| \llcorner N$, $N = \{x \in U \colon D_{\mu_V} \|\delta V\|(x) = \infty\}$, and

$$H_V(x) = D_{\mu_V} \|\delta V\|(x) \eta_V(x).$$

Hence we can write (3.27) as

$$\begin{aligned}\delta V(X) &= \int_{G_{n,m}(U)} \operatorname{div}_S X(x)\, dV(x,S) \\ &= -\int_U \langle X, H_V\rangle\, d\mu_V - \int_U \langle X, \eta_V\rangle\, d\sigma \\ &= -\int_U \langle X, H_V\rangle\, d\mu_V - \int_N \langle X, \eta_V\rangle\, d\sigma.\end{aligned}$$

We call H_V the *generalized mean curvature* of V, N the *generalized boundary* of V, σ the *generalized boundary measure* of V, and $\eta_V|N$ the *generalized unit co-normal* of V.

Remark 3.28. If $M \subset \mathbb{R}^n$ is a C^2-smooth m-dimensional submanifold with smooth boundary ∂M and $X \in C^1(U, \mathbb{R}^n)$, with $M \subset U,\ U \subset \mathbb{R}^n$ open, then

$$\int_M \operatorname{div}_M X\, d\mathcal{H}^m = -\int_M \langle X, H\rangle d\mathcal{H}^m - \int_{\partial M} \langle X, \eta\rangle d\mathcal{H}^{m-1},$$

where H is the mean curvature (vector) of M and η the inward pointing unit co-normal of ∂M, that is, $|\eta| \equiv 1$, η is normal to ∂M, tangential to M, and points inwards to M.

The first variation formula is applied with certain specific choices of the vector field X. Most importantly, we obtain the so-called monotonicity formula and its applications to the regularity theory of varifolds. These will be discussed in a series of presentations in home work sessions. If the time permits, we will return to these topics in context of currents.

4 Currents

In this Section we introduce and study some basic notions in the theory of currents which (like varifolds and m-rectifiable varifolds) are kind of generalized surfaces.

Let us start with the following motivating example.

Example 4.1. Let M be a smooth oriented m-dimensional submanifold of $\mathbb{R}^n$. We can integrate smooth differential m-forms α (with compact support) over M and thus consider M as a linear functional

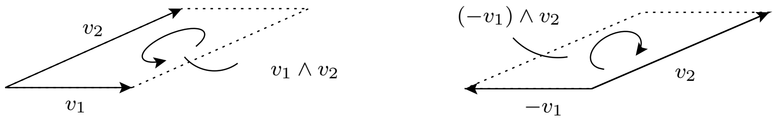
$$[M]\colon \{\text{smooth differential } m\text{-forms with compact support}\} \to \mathbb{R},$$

$$[M](\alpha) = \int_M \alpha.$$

Currents are, by definition, such continuous linear functionals on the space of smooth differential m-forms with compact support; see Definition 4.21.

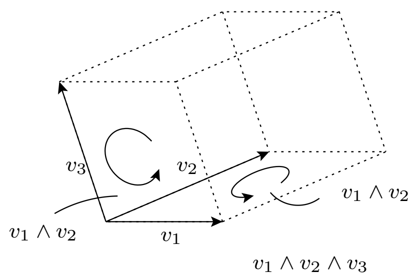
4.2 m-vectors

In this subsection, we discuss briefly about m-vectors which are kind of "products" of vectors. Given $v_1, v_2 \in \mathbb{R}^n$, a geometric interpretation of the 2-vector $v_1 \wedge v_2$ is the oriented parallelogram spanned by vectors v_1 and v_2.



If $v_1 = \lambda v_2$, the parallelogram is degenerate, and we have $v_1 \wedge v_2 = 0$.

Similarly, for a 3-vector $v_1 \wedge v_2 \wedge v_3$ can be interpreted as an oriented parallelepiped spanned by vectors $v_1, v_2, v_3 \in \mathbb{R}^n$.



Formally, the quickest (but not necessarily the most elegant) way to define the *vector space* of m-vectors

$$\bigwedge\nolimits_m(\mathbb{R}^n), \quad m = 0, \ldots, n,$$

is as the space of all (real) linear combinations

$$\sum_{1 \le i_1 < \cdots < i_m \le n} \underbrace{a_{i_1 \cdots i_m}}_{\in \mathbb{R}} e_{i_1} \wedge \cdots \wedge e_{i_m},$$

where $(e_1, \ldots, e_n)$ is the standard (ordered) basis of $\mathbb{R}^n$. The *basis (m-)vectors* $e_{i_1} \wedge \cdots \wedge e_{i_m}$, $1 \le i_1 < \cdots < i_m \le n$, of $\bigwedge_m(\mathbb{R}^n)$ can be defined as the strictly increasing sequences $i_1 < \cdots < i_m$. Thus we may identify $e_{i_1} \wedge \cdots \wedge e_{i_m}$ with the m-tuple $(i_1, \ldots, i_m)$ if $i_1 < \cdots < i_m$. Hence

$$\dim \bigwedge\nolimits_m(\mathbb{R}^n) = \binom{n}{m}.$$

If $m = n$, $e_1 \wedge \cdots \wedge e_n$ is the only basis vector, and therefore

$$\dim \bigwedge\nolimits_n(\mathbb{R}^n) = 1.$$

Hence we may identify

$$\bigwedge\nolimits_n(\mathbb{R}^n) = \mathbb{R}.$$

Similarly,

$$\bigwedge\nolimits_1(\mathbb{R}^n) = \operatorname{span}(e_1, \ldots, e_n) = \mathbb{R}^n.$$

We also define

$$\bigwedge\nolimits_0(\mathbb{R}^n) = \mathbb{R} \quad \text{and} \quad \bigwedge\nolimits_k(\mathbb{R}^n) = \{0\} \text{ for } k > n.$$

We want to "multiply" k-vectors and m-vectors and hence to give $e_{i_1} \wedge \cdots \wedge e_{i_m}$ a meaning as a "product" of vectors $e_{i_1}, \ldots, e_{i_m}$. Since the desired properties of the *wedge product* (or *exterior product*) $\bigwedge_k(\mathbb{R}^n) \times \bigwedge_m(\mathbb{R}^n) \to \bigwedge_{k+m}(\mathbb{R}^n)$ are

(a) *multilinearity*:

$$(au+bv)\wedge cw = ac(u\wedge w)+bc(v\wedge w), \quad a,b,c\in\mathbb{R},\ u,v\in \bigwedge{}_k(\mathbb{R}^n),\ w\in\bigwedge{}_m(\mathbb{R}^n);$$
$$au\wedge(bv+cw) = ab(u\wedge v)+ac(u\wedge w), \quad a,b,c\in\mathbb{R},\ u\in \bigwedge{}_k(\mathbb{R}^n),\ v,w\in\bigwedge{}_m(\mathbb{R}^n),$$

(b) *associativity*:

$$u\wedge(v\wedge w)=(u\wedge v)\wedge w, \quad \text{and}$$

(c) *anticommutativity*:

$$u\wedge v = (-1)^{km} v\wedge u, \quad u\in \bigwedge{}_k(\mathbb{R}^n),\ v\in\bigwedge{}_m(\mathbb{R}^n),$$

it is enough to define wedge products

$$e_{i_1}\wedge e_{i_2}\wedge\cdots\wedge e_{i_m}\wedge e_j = (e_{i_1}\wedge e_{i_2}\wedge\cdots\wedge e_{i_m})\wedge e_j$$

for $1\le i_1<\cdots<i_m$ and $j\in\{1,\ldots,n\}$. We have already defined $e_i\wedge e_j\in\bigwedge_2(\mathbb{R}^n)$ for $1\le i<j\le n$ (as the oriented pair (i,j) or the positively oriented unit square in $\mathbb{R}^n$ spanned by e_i and e_j). We define

$$e_j\wedge e_i = -e_i\wedge e_j \quad \text{for } i<j,$$

and

$$e_i\wedge e_i = 0.$$

We also have defined already $e_i\wedge e_j\wedge e_k$ if $i<j<k$, i.e. the positively oriented unit cube spanned by e_i, e_j, e_k (or the oriented 3-tuple (i,j,k)). So, we define for $i<j<k$

$$\begin{aligned}
e_i\wedge e_k\wedge e_j &= e_i\wedge(e_k\wedge e_j)\\
&= e_i\wedge(-e_j\wedge e_k)\\
&= -e_i\wedge e_j\wedge e_k,\\
e_k\wedge e_i\wedge e_j &= -e_i\wedge e_k\wedge e_j)\\
&= -(-e_i\wedge e_j\wedge e_k)\\
&= e_i\wedge e_j\wedge e_k,
\end{aligned}$$

and so on. Also $e_i\wedge e_i\wedge e_k=0$ if $i=j$ or $i=k$, or $j=k$. Continuing this way we have the wedge product $u\wedge v\in\bigwedge_{k+m}(\mathbb{R}^n)$ for $u\in\bigwedge_k(\mathbb{R}^n)$ and $v\in\bigwedge_m(\mathbb{R}^n)$. If $k+m>n$, $u\wedge v=0$ and $\bigwedge_{k+m}(\mathbb{R}^n)=\{0\}$. Let us summarize the discussion above:

Proposition 4.3. *The wedge product has the properties:*

(a) multilinearity*:*

(4.4) $$(au+bv)\wedge cw = ac(u\wedge w)+bc(v\wedge w)$$

for $a,b,c\in\mathbb{R},\ u,v\in\bigwedge_k(\mathbb{R}^n),\ w\in\bigwedge_m(\mathbb{R}^n),$

(b) associativity*:*

(4.5) $$u\wedge(v\wedge w)=(u\wedge v)\wedge w,$$

and

(c) anticommutativity*:*

$$u \wedge v = (-1)^{km} v \wedge u \tag{4.6}$$

for $u \in \bigwedge_k(\mathbb{R}^n),\ v \in \bigwedge_m(\mathbb{R}^n)$.

Since the m-vectors $e_{i_1} \wedge \cdots \wedge e_{i_m}$, $1 \leq i_1 < \cdots < i_m \leq n$, form a basis of $\bigwedge_m(\mathbb{R}^n)$, we may equip $\bigwedge_m(\mathbb{R}^n)$ with an inner product $\langle \cdot, \cdot \rangle$ such that these m-vectors form an orthonormal basis. More precisely, denote

$$\bigwedge(n,m) = \{(i_1, \ldots, i_m) \in \mathbb{N}^m \colon 1 \leq i_1 < \cdots < i_m \leq n\}$$

and $e_I = e_{i_1} \wedge \cdots \wedge e_{i_m}$ for $I = (i_1, \ldots, i_m) \in \bigwedge(n,m)$. Then

$$\left\langle \sum_{I \in \bigwedge(n,m)} a_I e_I, \sum_{J \in \bigwedge(n,m)} b_J e_J \right\rangle = \sum_{I \in \bigwedge(n,m)} a_I b_I. \tag{4.7}$$

In fact, identifying $\bigwedge_m(\mathbb{R}^n)$ and $\mathbb{R}^{\binom{n}{m}}$ isomorphically, i.e. by identifying the basis vectors e_I, $I \in \bigwedge(n,m)$, with the standard basis vectors of $\mathbb{R}^{\binom{n}{m}}$, the inner product in (4.7) becomes the standard inner product in $\mathbb{R}^{\binom{n}{m}}$.

We define the norm

$$|v| = \sqrt{\langle v, v \rangle} \tag{4.8}$$

for $v \in \bigwedge_m(\mathbb{R}^n)$. If v is a *simple* m-vector, that is

$$v = v_1 \wedge \cdots \wedge v_m$$

for some vectors $v_1, \ldots, v_m \in \mathbb{R}^n$, then

$$|v| = |v_1 \wedge \cdots \wedge v_m| \tag{4.9}$$

is the (m-dimensional) volume of the parallelepiped spanned by $v_1, \ldots, v_m$. In particular,

$$|v_1 \wedge \cdots \wedge v_m| = 0$$

if and only if $v_1, \ldots, v_m$ are linearly dependent.

4.10 m-covectors

Let $\bigwedge^1(\mathbb{R}^n)$ denote the dual of $\mathbb{R}^n$ (thus $\bigwedge^1(\mathbb{R}^n) = (\mathbb{R}^n)^*$) and let $dx^1, \ldots, dx^n$ denote the dual basis of $e_1, \ldots, e_n$. That is,

$$dx^i(e_j) = \delta^i_j = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{if } i \neq j. \end{cases}$$

Then we define the vector space

$$\bigwedge^m(\mathbb{R}^n) = \bigwedge_m\big(\bigwedge^1(\mathbb{R}^n)\big) \tag{4.11}$$

as above by replacing e_i with dx^i. The elements

$$\alpha = \sum_{i_1 < \cdots < i_m} a_{i_1 \cdots i_m} dx^{i_1} \wedge \cdots \wedge dx^{i_m} = \sum_{I \in \bigwedge(n,m)} a_I dx^I \tag{4.12}$$

of $\bigwedge^m(\mathbb{R}^n)$ are called *m-covectors*. The space $\bigwedge^m(\mathbb{R}^n)$ has the induced) inner product

$$\left\langle \sum_{I\in\bigwedge(n,m)} a_I dx^I, \sum_{J\in\bigwedge(n,m)} b_J dx^J \right\rangle = \sum_{I\in\bigwedge(n,m)} a_I b_I$$

such that the m-covectors $dx^{i_1} \wedge \cdots \wedge dx^{i_m}$, $1 \le i_1 < \cdots < i_m \le n$, form an orthonormal basis. Moreover, $\bigwedge^m(\mathbb{R}^n)$ is the dual vector space of $\bigwedge_m(\mathbb{R}^n)$. Again we have $\bigwedge^0(\mathbb{R}^n) = \mathbb{R} = \bigwedge^n(\mathbb{R}^n)$, $\bigwedge^1(\mathbb{R}^n) = \mathbb{R}^n$, and $\bigwedge^m(\mathbb{R}^n) = \{0\}$ if $m > n$.

4.13 m-vector fields, m-covector fields, and smooth differential m-forms

Definition 4.14. If $U \subset \mathbb{R}^n$, the mappings $U \to \bigwedge_m(\mathbb{R}^n)$,

$$x \mapsto \sum_{I\in\bigwedge(n,m)} a_I(x) e_I,$$

and $U \to \bigwedge^m(\mathbb{R}^n)$,

$$x \mapsto \sum_{I\in\bigwedge(n,m)} a_I(x) dx^I,$$

are called m-vector fields and m-covector fields in U, respectively.

Definition 4.15. The mappings $U \to \bigwedge^m(\mathbb{R}^n)$,

$$x \mapsto \sum_{I\in\bigwedge(n,m)} a_I(x) dx^I,$$

are also called (differential) m-forms in U.

If $U \subset \mathbb{R}^n$ is open and

$$\alpha = \sum_{I\in\bigwedge(n,m)} \alpha_I(x) dx^I,$$

where the functions α_I are C^∞-smooth, we say that α is a C^∞-smooth differential m-form in U. The space of all C^∞-smooth differential m-forms in U will be denoted by $\mathcal{A}^m(U)$.

Since $\bigwedge^0(\mathbb{R}^n) = \mathbb{R}$, we have $\mathcal{A}^0(U) = C^\infty(U,\mathbb{R})$. If $f\colon U \to \mathbb{R}$ is C^∞, i.e. $f \in \mathcal{A}^0(U)$, its *differential* $df\colon U \to \bigwedge^1(\mathbb{R}^n)$ is a C^∞-smooth differential 1-form such that at a point $x \in U$, $df(x)\colon \mathbb{R}^n \to \mathbb{R}$ is the linear mapping defined by

$$df(x)v = \langle \nabla f(x), v\rangle, \quad v \in \mathbb{R}^n.$$

Since, on the other hand,

$$dx^i(v) = dx^i\left(\sum_{j=1}^n v_j e_j\right) = v_i$$

and hence

$$\begin{aligned} df(x)v = \langle \nabla f(x), v\rangle &= \left\langle \sum_{i=1}^n \frac{\partial f(x)}{\partial x_i} e_i, v \right\rangle \\ &= \sum_{i=1}^n \frac{\partial f(x)}{\partial x_i} v_i = \sum_{i=1}^n \frac{\partial f(x)}{\partial x_i} dx^i(v), \end{aligned}$$

we notice that

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx^i. \tag{4.16}$$

Moreover, dx^i is the differential of the i^{th} coordinate function $x \mapsto x_i$.

Definition 4.17. Let

$$\alpha = \sum_{I \in \bigwedge(n,m)} \alpha_I dx^I$$

be a C^∞-smooth differential m-form. The *exterior derivative* of α is the $(m+1)$-form

$$d\alpha = \sum_{I \in \bigwedge(n,m)} d\alpha_I \wedge dx^I = \sum_{I \in \bigwedge(n,m)} \sum_{i=1}^{n} \frac{\partial \alpha_I}{\partial x_i} dx^i \wedge dx^I.$$

In particular, df is the exterior derivative of a 0-form f.

Using the facts that

$$\frac{\partial^2 \alpha_I}{\partial x_i \partial x_j} = \frac{\partial^2 \alpha_I}{\partial x_j \partial x_i}$$

and

$$dx^i \wedge dx^j = -dx^j \wedge dx^i,$$

we obtain

$$d^2\alpha = d(d\alpha) = 0.$$

Definition 4.18. Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^d$ be open sets and $f = (f^1, \ldots, f^d) \colon U \to V$ a C^∞-smooth mapping. The *pull-back* of a differential m-form α in V,

$$\alpha = \sum_{1 \le i_1 < \cdots < i_m \le d} \alpha_{i_1 \cdots i_m} dx^{i_1} \wedge \cdots \wedge dx^{i_m},$$

is the differential m-form $f^*\alpha$ in U defined by

$$f^*\alpha = \sum_{1 \le i_1 < \cdots < i_m \le d} \big(\alpha_{i_1 \cdots i_m} \circ f\big) df^{i_1} \wedge \cdots \wedge df^{i_m},$$

where

$$df^j = \sum_{i=1}^{n} \frac{\partial f^j}{\partial x_i} dx^i.$$

Notice that we do not require α being smooth. The pull-back and the exterior derivative commute, that is

$$f^*(d\alpha) = df^*\alpha \tag{4.19}$$

for smooth α.

Let $\mathcal{D}^m(U) \subset \mathcal{A}^m(U)$ denote the space of all C^∞-smooth differential m-forms in U with compact support, that is, if

$$\alpha = \sum_I \alpha_I dx^I,$$

then each α_I is C^∞-smooth and there exists a compact set $K \subset U$ such that $\operatorname{supp} \alpha_I \subset K$ for every I. We endow $\mathcal{D}^m(U)$ with the locally convex topology by saying that a sequence $\alpha^k \in \mathcal{D}^m(U),\ k \in \mathbb{N}$,

$$\alpha^k = \sum_I \alpha_I^k dx^I$$

converges to

$$\alpha = \sum_I \alpha_I dx^I \in \mathcal{D}^m(U)$$

if there exists a compact set $K \subset U$ such that

$$\operatorname{supp} \alpha^k := \bigcup_I \operatorname{supp} \alpha_I^k \subset K \quad \forall k$$

and

$$\frac{\partial^{|J|}\alpha_I^k}{\partial x^J} \to \frac{\partial^{|J|}\alpha_I}{\partial x^J}$$

uniformly as $k \to \infty$ for every multi-index $J = i_1 \cdots i_\ell$.

4.20 m-currents; definition and basic notions

Definition 4.21. An *m-current* in an open set $U \subset \mathbb{R}^n$ is a continuous (w.r.t. the locally convex topology described above) linear functional

$$T \colon \mathcal{D}^m(U) \to \mathbb{R}.$$

The space of m-currents in U is denoted by $\mathcal{D}_m(U)$.

Definition 4.22. The *boundary* of an m-current $T \in \mathcal{D}_m(U)$ is the $(m-1)$-current $\partial T \in \mathcal{D}_{m-1}(U)$ defined by

$$\partial T(\omega) = T(d\omega)$$

for all $\omega \in \mathcal{D}^{m-1}(U)$. Since $d^2 = 0$, we have $\partial^2 T = \partial(\partial T) = 0$.

Example 4.23. Let $M \subset \mathbb{R}^n$ be a smooth oriented m-dimensional submanifold with smooth boundary ∂M. Let $U \subset \mathbb{R}^n$ be an open set such that $M \cup \partial M \subset U$. Then M and ∂M define currents

$$[M] \in \mathcal{D}_m(U) \colon [M](\omega) = \int_M \omega, \quad \omega \in \mathcal{D}^m(U),$$
$$[\partial M] \in \mathcal{D}_{m-1}(U) \colon [\partial M](\alpha) = \int_{\partial M} \alpha, \quad \alpha \in \mathcal{D}^{m-1}(U).$$

By Stokes' theorem

$$[\partial M](\alpha) = \int_{\partial M} \alpha = \int_M d\alpha = [M](d\alpha)$$

for all $\alpha \in \mathcal{D}^{m-1}(U)$. Hence $\partial[M] = [\partial M]$.

Remark 4.24. For the definitions of the integrals

$$\int_M d\alpha \quad \text{and} \quad \int_{\partial M} \alpha$$

we refer to literature on differential geometry (e.g. [Lee], [Ho2]). However, since we will later integrate differential m-forms over "oriented" m-rectifiable sets, we will explain below the meaning of $\int_M \omega$ even in this more general setting.

Let $V \in G(n,m)$ and let $L\colon \mathbb{R}^m \to V$ be a linear isometric isomorphism (the restriction to $\mathbb{R}^m$ of an orthogonal mapping $O\colon \mathbb{R}^n \to \mathbb{R}^n$). Now

$$\begin{aligned}\textstyle\bigwedge_m(V) &= \{\textstyle\sum a_{i_1\cdots i_m} v_{i_1}\wedge\cdots\wedge v_{i_m} : v_{i_j}\in V\}\\ &= \{\lambda(Le_1)\wedge\cdots\wedge(Le_m) : \lambda\in\mathbb{R}\}\end{aligned}$$

is 1-dimensional. Hence, if $v \in \bigwedge_m(V)$, with $|v| = 1$, then the only other $w \in \bigwedge_m(V)$, with $|w| = 1$, is $w = -v$. Let T_xM be the tangent space of M at x (here, first, M is an oriented m-dimensional smooth submanifold of $\mathbb{R}^n$). Then M being "oriented" means that we have chosen, for every $x \in M$, an m-vector $\vec{M}(x) \in \bigwedge_m(T_xM)$ such that $|\vec{M}(x)| \equiv 1$ and $x \mapsto \vec{M}(x) \in G(n,m)$ is continuous. We then define

$$\int_M \omega = \int_M \langle \vec{M}(x), \omega(x)\rangle d\mathcal{H}^m(x).$$

Here $\langle \vec{M}(x), \omega(x)\rangle = \omega\big(\vec{M}(x)\big)(x) \in \mathbb{R}$ is the "dual pairing". In the general case, M will be m-rectifiable, T_x^mM the approximate tangent space of M, and $\vec{M}$ will be replaced by a "Borel orientation".

Example 4.25. 1. $m = 0$: For $a \in \mathbb{R}^n$, let $[a] = \delta_a \in \mathcal{D}_0(\mathbb{R}^n)$,

$$[a](\varphi) = \varphi(a), \quad \varphi \in \mathcal{D}^0(\mathbb{R}^n).$$

2. $m = 1$: Let $\Gamma \subset \mathbb{R}^n$ be a C^1-curve, $\vec{\Gamma}(x)$ the unit tangent vector to Γ such that $x \mapsto \vec{\Gamma}(x)$ is continuous. Then

$$[\Gamma](\omega) = \int_\Gamma \langle \vec{\Gamma}(x), \omega(x)\rangle d\mathcal{H}^1(x), \quad \omega \in \mathcal{D}^1(\mathbb{R}^n).$$

3. $m = n$: Let $U \subset \mathbb{R}^n$ be open with smooth boundary ∂U. Then

$$[U](\omega) = \int_U \langle e_1\wedge\cdots\wedge e_n, \omega(x)\rangle dm_n(x), \quad \omega \in \mathcal{D}^n(\mathbb{R}^n).$$

4. Let $Q = [0,1]\times[0,1]$ and let $T \in \mathcal{D}_1(\mathbb{R}^2)$ be defined as

$$T(\omega) = \int_Q \langle e_1, \omega(x)\rangle dm_2(x), \quad \omega \in \mathcal{D}^1(\mathbb{R}^2).$$

Writing $\omega = \omega_1 dx^1 + \omega_2 dx^2$, we see that

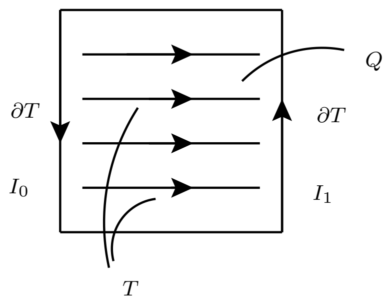
$$\begin{aligned}T(\omega) &= \int_Q \langle e_1, \omega(x)\rangle dm_2(x)\\ &= \int_Q \omega_1 \underbrace{\langle e_1, dx^1\rangle}_{\equiv 1} dm_2(x) + \int_Q \omega_2 \underbrace{\langle e_1, dx^2\rangle}_{\equiv 0} dm_2(x)\\ &= \int_Q \omega_1 dm_2(x).\end{aligned}$$

On the other, $\partial T \in \mathcal{D}_0(\mathbb{R}^2)$, so it operates on smooth functions (0-forms) as

$$\begin{aligned}\partial T(\varphi) = T(d\varphi) &= T\left(\frac{\partial \varphi}{\partial x_1} dx^1 + \frac{\partial \varphi}{\partial x_2} dx^2\right)\\ &= \int_Q \frac{\partial \varphi}{\partial x_1} dm_2\\ &= \int_0^1 \int_0^1 \frac{\partial \varphi}{\partial x_1} dx_1\, dx_2\\ &= \int_0^1 (\varphi(1, x_2) - \varphi(0, x_2))\, dx_2\\ &= \int_{I_1} \varphi - \int_{I_0} \varphi,\end{aligned}$$

where I_0 and I_1 are the line segments $I_0 = \big[(0,0),(0,1)\big]$, $I_1 = \big[(1,0),(1,1)\big]$. So,

$$\partial T = \mathcal{H}^1 \llcorner I_1 - \mathcal{H}^1 \llcorner I_0.$$



Notice that T is a 1-dimensional current but its "support" is 2-dimensional.

Definition 4.26. We define the *mass* of $T \in \mathcal{D}_m(U)$ by

(4.27) $$\mathbf{M}(T) = \sup\{T(\omega) \colon \omega \in \mathcal{D}^m(U), |\omega(x)| \le 1\ \forall x \in U\}.$$

If $W \subset U$ is open, we define

$$\mathbf{M}_W(T) = \sup\{T(\omega) \colon \omega \in \mathcal{D}^m(U), |\omega(x)| \le 1\ \forall x \in W,\ \operatorname{supp} \omega \subset W\}.$$

Remark 4.28. (a) There is another slightly different definition of the mass: Indeed, one first define the *co-mass* of an m-covector $\eta \in \bigwedge^m(\mathbb{R}^n)$ by

$$\|\eta\| = \sup\{\langle \zeta, \eta\rangle \colon |\zeta| \le 1,\ \zeta \in \bigwedge\nolimits_m(\mathbb{R}^n) \text{ simple}\}$$

and then

$$M(T) = \sup\{T(\omega) \colon \omega \in \mathcal{D}^m(U),\ \|\omega(x)\| \le 1\ \forall x \in U\}.$$

Since $\|\omega(x)\| \le |\omega(x)|$, it is possible that $M(T) > \mathbf{M}(T)$.

(b) Suppose that $L \colon \mathcal{D}^m(U) \to \mathbb{R}$ is a linear map that is continuous with respect to the norm topology of $\mathcal{D}^m(U)$, that is $L(\omega_i) \to L(\omega)$ if $\omega_i, \omega \in \mathcal{D}^m(U)$, with $|\omega_i - \omega| \to 0$. Since the convergence in the locally convex topology of $\mathcal{D}^m(U)$ implies the convergence in the norm topology, we notice that L is continuous with respect to the locally convex topology, too.

Hence $L \in \mathcal{D}_m(U)$. On the other hand, since the norm topology of $\mathcal{D}^m(U)$ is coarser than the locally convex topology, there can be m-currents with infinite mass. In other words, each m-current $T \in \mathcal{D}_m(U)$ is a linear mapping $T : \mathcal{D}_m(U) \to \mathbb{R}$ that is continuous with respect to the locally convex topology but not necessarily with respect to the norm topology of $\mathcal{D}^m(U)$.

(c) Since $(\mathcal{D}^m(U), |\cdot|)$ is a normed space, its dual space $\{T \in \mathcal{D}_m(U) \colon \mathbf{M}(T) < \infty\}$ is a Banach space.

Applying the Hahn-Banach theorem and the Riesz representation theorem we obtain the following:

Theorem 4.29. *Suppose that $T \in \mathcal{D}_m(U)$ such that $\mathbf{M}_W(T) < \infty$ for every $W \Subset U$. Then there exists a Radon measure μ_T on $\mathbb{R}^n$ and a μ_T-measurable mapping $\vec{T} \colon \mathbb{R}^n \to \bigwedge_m(\mathbb{R}^n)$ such that $|\vec{T}(x)| = 1$ for μ_T-a.e. $x \in \mathbb{R}^n$ and*

$$T(\omega) = \int \langle \vec{T}(x), \omega(x)\rangle d\mu_T(x) \quad \forall \omega \in \mathcal{D}^m(U).$$

The total variation measure *μ_T associated to T is characterized by*

$$\mu_T(W) = \sup\{T(\omega) \colon \omega \in \mathcal{D}^m(U),\ |\omega| \le 1,\ \operatorname{supp}\omega \subset W\} = \mathbf{M}_W(T)$$

for every open $W \subset U$. In particular,

$$\mu_T(U) = \mu_T(\mathbb{R}^n) = \mathbf{M}(T).$$

Definition 4.30 (Restrictions of currents)**.** If $T \in \mathcal{D}_m(U)$, $\mathbf{M}(T) < \infty$, and $A \subset \mathbb{R}^n$ is Borel, then the *restriction* of T to A is the m-current $T \llcorner A \in \mathcal{D}_m(U)$,

$$(T \llcorner A)(\omega) = \int_A \langle \vec{T}(x), \omega(x)\rangle d\mu_T(x), \quad \omega \in \mathcal{D}^m(U),$$

where $\vec{T}$ and μ_T are as in Theorem 4.29. Similarly, if g is a μ_T-integrable function, we define $T \llcorner g \in \mathcal{D}_m(U)$, the *interior multiplication by* g, by

$$(T \llcorner g)(\omega) = \int g(x)\langle \vec{T}(x), \omega(x)\rangle d\mu_T(x), \quad \omega \in \mathcal{D}^m(U).$$

Definition 4.31. The *support* of $T \in \mathcal{D}_m(U)$ is the set

$$\operatorname{supp} T = U \setminus \bigcup\{V \colon V \subset \mathbb{R}^n \text{ open},\ T(\omega) = 0\ \forall \omega \in \mathcal{D}^m(U),\ \operatorname{supp}\omega \subset V\}.$$

If $\mathbf{M}(T) < \infty$, and hence μ_T exists, then $\operatorname{supp} T = U \cap \operatorname{supp}\mu_T$. Recall that the support of the measure μ_T is the set

$$\operatorname{supp}\mu_T = \mathbb{R}^n \setminus \bigcup\{V \colon V \subset \mathbb{R}^n \text{ open}, \mu_T(V) = 0\}.$$

Definition 4.32. Let $T_i, T \in \mathcal{D}_m(U)$. We say that the sequence T_i converges to T and write

$$T_i \to T$$

if

$$\lim_{i\to\infty} T_i(\omega) = T(\omega)$$

for every $\omega \in \mathcal{D}^m(U)$. Hence, in fact, $T_i \xrightarrow{w^\star} T$.

Proposition 4.33. *Suppose $T_i \to T$. Then*

$$\partial T_i \to \partial T \quad \textit{and} \quad \mathbf{M}(T) \le \liminf_{i\to\infty} \mathbf{M}(T_i).$$

Remark 4.34. The lower semicontinuity of the mass is very important and useful property in mass minimizing problem.

Remark 4.35. We notice that the normed space $(\mathcal{D}^m(U), |\cdot|)$ is separable, and hence the closed unit ball of its dual $\{T \in \mathcal{D}_m(U) \colon \mathbf{M}(T) < \infty\}$ is sequentially compact in the weak* topology by the (sequential) Banach-Anaoglu theorem.

By applying the (sequential) Banach-Alaoglu theorem for the Banach space $\{T \in \mathcal{D}_m(U) \colon \mathbf{M}(T) < \infty\}$ we obtain the following:

Theorem 4.36. *Let $T_i \in \mathcal{D}_m(U)$ with*

$$\sup_i \mathbf{M}(T_i) < \infty.$$

Then there exist a subsequence T_{i_j} and $T \in \mathcal{D}_m(U)$ such that

$$T_{i_j} \to T.$$

Next we define the cartesian product of currents $T_i \in \mathcal{D}_{m_i}(U_i)$, $U_i \subset \mathbb{R}^{n_i}$ open, $i = 1, 2$. Any differential $(m_1 + m_2)$-form ω in $U_1 \times U_2$ can be written in the form

$$\omega(x, y) = \sum_{\substack{(I,J) \in \bigwedge(n_1, m_1') \times \bigwedge(n_2, m_2') \\ m_1' + m_2' = m_1 + m_2}} \omega_{IJ} dx^I \wedge dy^J, \quad (x, y) \in U_1 \times U_2.$$

Then we define:

Definition 4.37. Let $T_i \in \mathcal{D}_{m_i}(U_i)$, $U_i \subset \mathbb{R}^{n_i}$ open, $i = 1, 2$. The cartesian product $T_1 \times T_2 \in \mathcal{D}_{m_1+m_2}(U_1 \times U_2)$ is defined by

$$(T_1 \times T_2)(\omega) = T_1 \left(\sum_{I \in \bigwedge(n_1, m_1)} T_2 \left(\sum_{J \in \bigwedge(n_2, m_2)} \omega_{IJ} dy^J \right) dx^I \right)$$

for $\omega \in \mathcal{D}^{m_1+m_2}(U_1 \times U_2)$.

Notice that $T_1 \times T_2$ ignores the terms $dx^I \wedge dy^J$, where $I \in \bigwedge(n_1, m_1')$, $J \in \bigwedge(n_2, m_2')$, $m_1' + m_2' = m_1 + m_2$ but $(m_1', m_2') \ne (m_1, m_2)$.

Remark 4.38. (a) The motivation of the definition is, of course, that we want

$$[M_1 \times M_2] = [M_1] \times [M_2]$$

if M_1 and M_2 are smooth submanifolds.

(b) Since

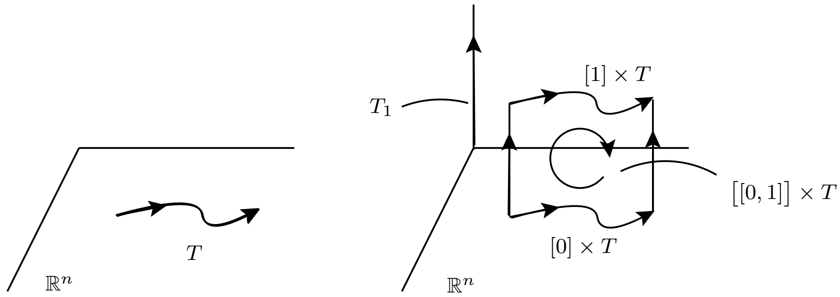
$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^m \alpha \wedge d\beta$$

for m-forms α, we have

$$\partial(T_1 \times T_2) = (\partial T_1) \times T_2 + (-1)^{m_1} T_1 \times (\partial T_2).$$

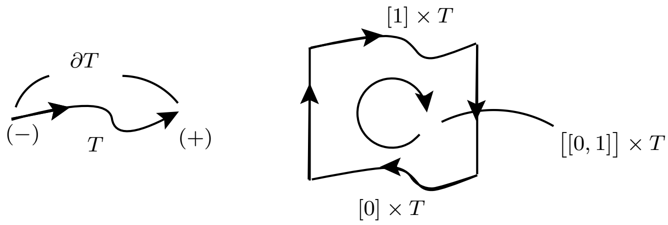
As an important special case we consider the following example:

Example 4.39. Let $T_1 = [[0,1]] \in \mathcal{D}_1(\mathbb{R})$ and $T_2 = T \in \mathcal{D}_m(\mathbb{R}^n)$.



Then

$$\begin{aligned}\partial\big([[0,1]] \times T\big) &= (\partial T_1) \times T - T_1 \times \partial T \\ &= \big([1] - [0]\big) \times T - T_1 \times \partial T \\ &= [1] \times T - [0] \times T - [[0,1]] \times \partial T.\end{aligned}$$



Next we define the push-forward of a current under a smooth mapping.

Definition 4.40. Suppose that $U\colon \mathbb{R}^n$ and $V \subset \mathbb{R}^d$ are open sets and $f\colon U \to V$ a C^∞-mapping. Let $T \in \mathcal{D}_m(U)$ be such that $f|\operatorname{supp} T$ is proper, i.e. $f^{-1}K \cap \operatorname{supp} T$ is compact for every compact $K \subset V$. We define $f_\sharp T \in \mathcal{D}_m(V)$, the *push-forward* of T under f, by

$$f_\sharp T(\omega) = T(\varphi f^* \omega), \quad \omega \in \mathcal{D}_m(V),$$

where $\varphi \in C_0^\infty(U)$ is any function such that $\varphi \equiv 1$ in the compact set $\operatorname{supp} T \cap f^{-1} \operatorname{supp} \omega \subset U$.

Notice that $\varphi f^* \omega \in \mathcal{D}^m(U)$ but it is possible that $f^*\omega \notin \mathcal{D}^m(U)$ since $\operatorname{supp} f^*\omega$ need not be compact.

Remark 4.41.

1. If f and T are as above, then $\partial f_\sharp T = f_\sharp \partial T$.
2. If $T_i \to T$ and $f|(\operatorname{supp} T_i \cup \operatorname{supp} T)$ is proper, then $f_\sharp T_i \to f_\sharp T$.
3. Suppose that $\mathbf{M}_W(T) < \infty$ for every $W \Subset U$, and hence

   $$T(\omega) = \int \langle \vec{T}(x), \omega(x) \rangle d\mu_T(x) \quad \forall \omega \in \mathcal{D}^m(U),$$

   where $\vec{T}$ and μ_T are given by Theorem 4.29. Then the push-forward $f_\sharp T$ is given by

   $$f_\sharp T(\omega) = \int \langle f^*\omega, \vec{T} \rangle d\mu_T = \int \big\langle \omega\big(f(x)\big), \textstyle\bigwedge_m df_x \vec{T}(x) \big\rangle d\mu_T(x).$$

Notice that the formula makes sense if f is C^1, with $f|\operatorname{supp} T$ proper. Above $\bigwedge_m df_x$ is the linear map $\bigwedge_m df_x \colon \bigwedge_m(\mathbb{R}^n) \to \bigwedge_m(\mathbb{R}^d)$ defined by

$$\bigwedge{}_m df_x(e_{i_1} \wedge \cdots \wedge e_{i_m}) = df_x(e_{i_1}) \wedge \cdots \wedge df_x(e_{i_m})$$

for every $(i_1, \ldots, i_m) \in \bigwedge(n, m)$.

Now we can define the *homotopy formula* for currents. For that purpose let $V \subset \mathbb{R}^d$ be open and let $f, g \colon U \to V$ be smooth mappings. Furthermore, suppose that $h \colon [0,1] \times U \to V$ is smooth such that

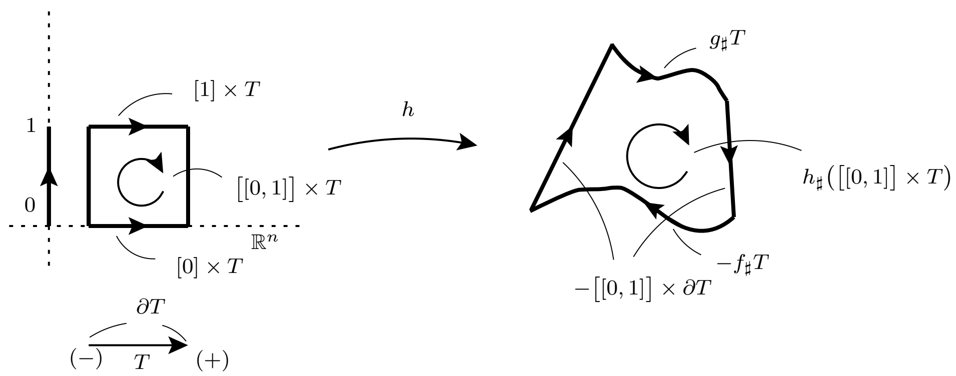
$$h(0, x) = f(x) \quad \text{and} \quad h(1, x) = g(x) \quad \forall x \in U.$$

Since (see Example 4.39)

$$\begin{aligned} \partial h_\sharp\big([\![0,1]\!] \times T\big) &= h_\sharp \partial\big([\![0,1]\!] \times T\big) \\ &= h_\sharp\big([1] \times T - [0] \times T - [\![0,1]\!] \times \partial T\big) \\ &= h_\sharp\big([1] \times T\big) - h_\sharp\big([0] \times T\big) - h_\sharp\big([\![0,1]\!] \times \partial T\big) \\ &= g_\sharp T - f_\sharp T - h_\sharp\big([\![0,1]\!] \times \partial T\big), \end{aligned}$$

we have

$$g_\sharp T - f_\sharp T = \partial h_\sharp\big([\![0,1]\!] \times T\big) + h_\sharp\big([\![0,1]\!] \times \partial T\big). \tag{4.42}$$



An important special case is the *affine homotopy*

$$h(t, x) = tg(x) + (1 - t)f(x).$$

Definition 4.43 (Cone)**.** Let $T \in \mathcal{D}_m(U)$ with $\operatorname{supp} T$ compact. The *cone over* T is

$$0 \lhd T = h_\sharp\big([\![0,1]\!] \times T\big) \in \mathcal{D}_{m+1}(\mathbb{R}^n),$$

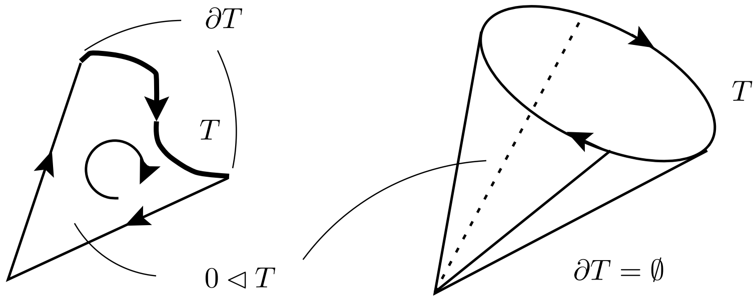
where $h(t, x) = tx$.

We notice that

$$\partial(0 \lhd T) = T - 0 \lhd \partial T.$$

In particular, if T has no boundary, then T itself is a boundary:

$$T = \partial(0 \lhd T).$$



For a linear mapping $L\colon \mathbb{R}^n \to \mathbb{R}^d$ we denote by $\bigwedge_m L$ the linear mapping

$$\bigwedge{}_m L\colon \bigwedge{}_m(\mathbb{R}^n) \to \bigwedge{}_m(\mathbb{R}^d)$$

defined by

$$\bigwedge{}_m L(e_{i_1} \wedge \cdots \wedge e_{i_m}) = Le_{i_1} \wedge \cdots \wedge Le_{i_m}$$

for every $(i_1, \dots, i_m) \in \bigwedge(n, m)$. If $f\colon U \to V$ is smooth ($V \subset \mathbb{R}^d$ open), we see that

$$\langle v, f^*\omega(x)\rangle = \big\langle \bigwedge{}_m df_x(v), \omega(x)\big\rangle$$

for all $v \in \bigwedge_m(\mathbb{R}^n)$, $\omega \in \mathcal{D}^m(V)$, and $x \in U$. Hence we can state:

Proposition 4.44. *If $T \in \mathcal{D}_m(U)$, with $\operatorname{supp} T$ compact and $\mathbf{M}(T) < \infty$ and if $f\colon U \to V$ is C^∞, with $f|\operatorname{supp} T$ proper, then*

$$f_\sharp T(\omega) = \int \big\langle \omega\big(f(x)\big), \bigwedge{}_m df_x \vec{T}(x)\big\rangle d\mu_T(x)$$

and

$$\mathbf{M}(f_\sharp T) \le \operatorname{Lip}(f|\operatorname{supp} T)^m \mathbf{M}(T). \tag{4.45}$$

Recall that

$$\operatorname{Lip}(g) := \sup\left\{\frac{|g(x) - g(y)|}{|x - y|} : x \neq y\right\}.$$

The inequality (4.45) follows from the estimate

$$\big|\bigwedge{}_m df_x\big(\vec{T}(x)\big)\big| \le \operatorname{Lip}(f|\operatorname{supp} T)^m, \quad x \in \operatorname{supp} T,$$

which, in turn, is a consequence of

$$\big|\bigwedge{}_m L(e_{i_1} \wedge \cdots \wedge e_{i_m})\big| \le \|L\|^m.$$

Suppose that $h\colon [0,1] \times U \to V$ is the affine homotopy $h(t,x) = tg(x) + (1-t)f(x)$ between smooth mappings $f, g\colon U \to V$. If $T \in \mathcal{D}_m(U)$, with $\mathbf{M}(T) < \infty$, we have

$$\mathbf{M}\big(h_\sharp\big([\![0,1]\!] \times T\big)\big) \le \sup_{\operatorname{supp} T} |f - g| \sup_{x \in \operatorname{supp} T} \big(|df_x| + |dg_x|\big)^m \mathbf{M}(T). \tag{4.46}$$

This follows from the integral representation (Theorem 4.29) since

$$\overrightarrow{[\![0,1]\!] \times T} = e_1 \wedge \vec{T} \quad \text{and} \quad \mu_{[\![0,1]\!] \times T} = (m_1 \llcorner [0,1]) \times \mu_T,$$

and therefore

$$\begin{aligned} h_\sharp\big([\![0,1]\!]\times T\big)(\omega) &= \int \langle \omega\big(h(t,x)\big), \textstyle\bigwedge_{m+1} dh_{(t,x)}\big(e_1 \wedge \vec{T}(x)\big)\rangle d\mu_{[\![0,1]\!]\times T} \\ &= \int_0^1 \Big(\int \langle \omega\big(h(t,x)\big), \big(g(x)-f(x)\big)e_1 \wedge \textstyle\bigwedge_m (tdg_x + (1-t)df_x)\vec{T}(x)\rangle d\mu_T \Big)\, dt. \end{aligned}$$

Next we state a couple of further consequences of the homotopy formula.

Lemma 4.47. *Let $T \in \mathcal{D}_m(U)$, with $\mathbf{M}_W(T) < \infty$ and $\mathbf{M}_W(\partial T) < \infty$ for every $W \Subset U$. If $f, g\colon U \to V \subset \mathbb{R}^d$ are C^1 smooth with $f|\operatorname{supp} T = g|\operatorname{supp} T$ proper, then $f_\sharp T = g_\sharp T$.*

Proof. Applying the homotopy formula (4.42) with $h(t,x) = tg(x) + (1-t)f(x)$ we obtain

$$\begin{aligned} g_\sharp T(\omega) - f_\sharp T(\omega) &= \partial h_\sharp\big([\![0,1]\!]\times T\big)(\omega) + h_\sharp\big([\![0,1]\!]\times \partial T\big)(\omega) \\ &= h_\sharp\big([\![0,1]\!]\times T\big)(d\omega) + h_\sharp\big([\![0,1]\!]\times \partial T\big)(\omega), \end{aligned}$$

and therefore, by (4.46),

$$\begin{aligned} \big|g_\sharp T(\omega) - f_\sharp T(\omega)\big| &= \big|h_\sharp\big([\![0,1]\!]\times T\big)(d\omega) + h_\sharp\big([\![0,1]\!]\times \partial T\big)(\omega)\big| \\ &\le \big|h_\sharp\big([\![0,1]\!]\times T\big)(d\omega)\big| + \big|h_\sharp\big([\![0,1]\!]\times \partial T\big)(\omega)\big| \\ &\le \mathbf{M}\big(h_\sharp\big([\![0,1]\!]\times T\big)\big)|d\omega| + \mathbf{M}\big(h_\sharp\big([\![0,1]\!]\times \partial T\big)\big)|\omega| \\ &\le c\big[\mathbf{M}(T)|d\omega| + \mathbf{M}(\partial T)|\omega|\big] \sup_{\operatorname{supp} T} |g-f| = 0 \end{aligned}$$

since, by assumption, $f = g$ in $\operatorname{supp} T$. □

With help of the homotopy formula we can define $f_\sharp T$ for a Lipschitz mapping $f\colon U \to V \subset \mathbb{R}^d$ provided $f|\operatorname{supp} T$ is proper and $\mathbf{M}_W(T) < \infty$, $\mathbf{M}_W(T) < \infty$ for every $W \Subset U$. For that purpose, let η_ε, $\varepsilon > 0$, be a standard mollifier;

$$\eta_\varepsilon(x) = \varepsilon^{-n}\eta(x/\varepsilon),$$

where $\eta\colon \mathbb{R}^n \to [0,\infty)$ is C^∞, with $\operatorname{supp}\eta \subset B(0,1)$ and $\int \eta = 1$.

Given a Lipschitz map $f\colon U \to V$ we define the C^∞ mapping $f^{(\varepsilon)} = f * \eta_\varepsilon$.

Lemma 4.48. *Let $T \in \mathcal{D}_m(U)$, with $\mathbf{M}_W(T) < \infty$ and $\mathbf{M}_W(\partial T) < \infty$ for every $W \Subset U$. Let $f\colon U \to V \subset \mathbb{R}^d$ be Lipschitz with $f|\operatorname{supp} T$ proper. Then the limit*

$$f_\sharp T(\omega) := \lim_{\varepsilon\to 0} f_\sharp^{(\varepsilon)} T(\omega)$$

exists for every $\omega \in \mathcal{D}^m(V)$. Moreover, $\operatorname{supp} f_\sharp T \subset f(\operatorname{supp} T)$ *and*

$$\mathbf{M}_W(f_\sharp T) \le \big(\operatorname*{ess\,sup}_{f^{-1}W} |df_x|\big)^m \mathbf{M}_{f^{-1}W}(T)$$

for every $W \Subset V$.

Proof. Fix $\omega \in \mathcal{D}^m(V)$. If $\varepsilon > 0$ and $\sigma > 0$ are sufficiently small (depending on $\omega \in \mathcal{D}^m(V)$), the homotopy formula with $h(t,x) = tf^{(\varepsilon)}(x) + (1-t)f^{(\sigma)}(x)$ implies

$$\begin{aligned} f_\sharp^{(\varepsilon)} T(\omega) - f_\sharp^{(\sigma)} T(\omega) &= \partial h_\sharp\big([\![0,1]\!]\times T\big)(\omega) + h_\sharp\big([\![0,1]\!]\times \partial T\big)(\omega) \\ &= h_\sharp\big([\![0,1]\!]\times T\big)(d\omega) + h_\sharp\big([\![0,1]\!]\times \partial T\big)(\omega). \end{aligned}$$

For sufficiently small $\varepsilon > 0$ and $\sigma > 0$ we get from (4.46)

$$\left|f_\sharp^{(\varepsilon)}T(\omega) - f_\sharp^{(\sigma)}T(\omega)\right| \le c \sup_{f^{-1}K\cap \operatorname{supp} T} \left|f^{(\varepsilon)} - f^{(\sigma)}\right| \operatorname{Lip}(f)^m,$$

where $K \subset V$ is a compact set containing $\operatorname{supp}\omega$ in its interior. Since $f^{(\varepsilon)} \xrightarrow{\varepsilon\to 0} f$ uniformly on compact subsets of U, the claims follow. □

Theorem 4.49 (Constancy theorem)**.** *Let $U \subset \mathbb{R}^n$ be a domain (i.e. open and connected). If $T \in \mathcal{D}_n(U)$, with $\partial T = 0$ and $\mathbf{M}_W(T) < \infty$ for all $W \Subset U$, then there exists a constant c such that*

$$T = c[U],$$

that is

$$T\big(\varphi dx^1 \wedge \cdots \wedge dx^n\big) = c\int_U \varphi dm_n$$

for every $\varphi \in C_0^\infty(U)$.

Note that $m = n$ above.

Proof. By Theorem 4.29 there exist a Radon measure μ_T and a μ_T-measurable function $\sigma\colon U \to \{-1, 1\}$ such that

$$T(\omega) = \int \langle \omega(x), \sigma(x)e_1 \wedge \cdots \wedge e_n\rangle d\mu_T(x) = \int \sigma\varphi\, d\mu_T = \int \varphi\, d\mu_T^+ - \int \varphi\, d\mu_T^-$$

for every $\omega = \varphi dx^1 \wedge \cdots \wedge dx^n \in \mathcal{D}^n(U)$, where $\mu_T^+ = \mu_T \llcorner \{\sigma = 1\}$ and $\mu_T^- = \mu_T \llcorner \{\sigma = -1\}$. Let $\eta_\varepsilon,\ \varepsilon > 0$, be as above. Define

$$T_\varepsilon(\omega) := T(\eta_\varepsilon * \omega)$$

for $0 < \varepsilon < \operatorname{dist}(\operatorname{supp}\omega, \partial U)$ and for continuous n-forms $\omega \in C_0(U, \bigwedge^n(\mathbb{R}^n))$ with compact support in U. Here $\eta_\varepsilon * \omega = \eta_\varepsilon * \varphi\, dx^1 \wedge \cdots \wedge dx^n$ if $\omega = \varphi\, dx^1 \wedge \cdots \wedge dx^n$. We first observe that, for fixed $W \Subset U$ and $\varepsilon > 0$, the set

$$S = \{\eta_\varepsilon * \omega \colon \omega \in C_0(U, \textstyle\bigwedge^n(\mathbb{R}^n), \operatorname{supp}\omega \subset \bar{W}, \displaystyle\int_U |\omega| dm_n \le 1\}$$

is compact in $C_0(U, \bigwedge^n(\mathbb{R}^n))$ with respect to the norm $(|\cdot|)$ topology. Hence, by continuity of T also with respect to the norm topology, there exists a constant $c = c(T, W, \varepsilon)$ such that

$$|T_\varepsilon(\omega)| \le c\int_U |\omega| dm_n \tag{4.50}$$

for every $\omega \in C_0(U, \bigwedge^n(\mathbb{R}^n)$, with $\operatorname{supp}\omega \subset \bar{W}$. On the other hand,

$$T_\varepsilon(\omega) = T(\eta_\varepsilon * \omega) = \int \eta_\varepsilon * \varphi\, d\mu_T^+ - \int \eta_\varepsilon * \varphi\, d\mu_T^-$$

if $\omega = \varphi dx^1 \wedge \cdots dx^n,\ \varphi \in C_0(W)$. Applying the Riesz representation theorem to positive linear functionals

$$\varphi \mapsto \int \eta_\varepsilon * \varphi\, d\mu_T^\pm, \quad \varphi \in C_0(W),$$

we get Radon measures μ_ε^+ and μ_ε^- such that

$$\int \eta_\varepsilon * \varphi \, d\mu_T^+ = \int \varphi \, d\mu_\varepsilon^+ \quad \text{and} \quad \int \eta_\varepsilon * \varphi \, d\mu_T^- = \int \varphi \, d\mu_\varepsilon^- .$$

Hence, by (4.50),

$$\left| \int \varphi \, d\mu_\varepsilon^+ - \int \varphi \, d\mu_\varepsilon^- \right| = |T_\varepsilon(\omega)| \le c \int_U |\omega| dm_n = c \int_{\operatorname{supp} \varphi} |\varphi| dm_n ,$$

and therefore $\mu_\varepsilon^+, \mu_\varepsilon^- \ll m_n$. The Radon-Nikodym theorem then implies that there exists $g_\varepsilon \in L^1(m_n)$ such that

$$T_\varepsilon(\omega) = \int \varphi g_\varepsilon \, dm_n \tag{4.51}$$

for $\omega = \varphi dx^1 \wedge \cdots \wedge dx^n$, $\varphi \in C_0(W)$. On the other hand, since $\partial T = 0$ by assumption, we have

$$T_\varepsilon(d\omega) = T(\eta_\varepsilon * d\omega) = T\big(d(\eta_\varepsilon * \omega)\big) = \partial T(\eta_\varepsilon * \omega) = 0 \tag{4.52}$$

if $\omega \in C_0^1(U, \bigwedge^{n-1}(\mathbb{R}^n))$, with $\operatorname{supp} \omega \subset W$. Applying this to

$$\omega = \varphi dx^1 \wedge \cdots \wedge dx^{j-1} \wedge dx^{j+1} \wedge \cdots \wedge dx^n ,$$

for which

$$d\omega = (-1)^{j-1} \frac{\partial \varphi}{\partial x_j} dx^1 \wedge \cdots \wedge dx^n ,$$

we get

$$T_\varepsilon(d\omega) = (-1)^{j-1} \int \frac{\partial \varphi}{\partial x_j} g_\varepsilon \, dm_n = 0 \tag{4.53}$$

for all $\varphi \in C_0^1(W)$ and for all $j \in \{1, \dots, n\}$. It follows that the distributional gradient of g_ε vanishes m_n-a.e. and therefore $g_\varepsilon = c_\varepsilon$ m_n-a.e., where c_ε is a constant[1]. Letting then $\varepsilon \to 0$ and $W \nearrow U$, we obtain (by continuity of T)

$$T(\omega) = \lim_{\varepsilon \to 0} T_\varepsilon(\omega) = \lim_{\varepsilon \to 0} c_\varepsilon \int_U \varphi dm_n = c \int_U \varphi dm_n = c[U](\varphi)$$

for all $\omega = \varphi dx^1 \wedge \cdots \wedge dx^n \in \mathcal{D}^n(U)$, where the limit

$$\lim_{\varepsilon \to 0} c_\varepsilon = c$$

exists since the limit

$$\lim_{\varepsilon \to 0} T_\varepsilon(\omega) = \lim_{\varepsilon \to 0} c_\varepsilon \int_U \varphi dm_n$$

exists. □

Next we want to weaken the assumption $\partial T = 0$ to $\mathbf{M}(\partial T) < \infty$. Before we state the theorem (Theorem 4.65), which a generalization of the Constancy theorem, we first discuss about *functions of bounded variation*. We refer to e.g. [EG], [Si], [Ho3] for more details.

[1]This follows from Poincaré's inequality for $W^{1,1}$-functions.

Definition 4.54. Let $U \subset \mathbb{R}^n$ be open and $u \in L^1_{\mathrm{loc}}(U)$. Define

$$\int_U |Du| := \sup\left\{ \int_U u \operatorname{div} g \colon g = (g_1, \ldots, g_n) \in C^1_0(U; \mathbb{R}^n),\ |g| \le 1 \right\}.$$

Above $\int_U |Du|$ should be understood just as a notation (not an integral). Furthermore,

$$\operatorname{div} g = \sum_{i=1}^n \frac{\partial g_i}{\partial x_i}$$

is the usual divergence.

Example 4.55. (a) If $u \in C^1(U)$, then integration by parts implies that

$$\int_U u \operatorname{div} g = -\int_U \nabla u \cdot g \quad \forall g \in C^1_0(U; \mathbb{R}^n),$$

and so

$$\int_U |Du| = \int_U |\nabla u|.$$

(b) More generally, if u belongs to the Sobolev space $W^{1,1}_{\mathrm{loc}}(U)$, then again

$$\int_U |Du| = \int_U |\nabla u|,$$

where ∇u is the distributional gradient of u.

Definition 4.56. A function $u \in L^1_{\mathrm{loc}}(U)$ is said to have *bounded variation in U* if

$$\int_U |Du| < \infty.$$

We denote by $\mathrm{BV}(U)$ the vector space of all functions $u \in L^1(U)$ with bounded variation in U.

Definition 4.57. Similarly, a function $u \in L^1_{\mathrm{loc}}(U)$ has *locally bounded variation* and belongs to $\mathrm{BV}_{\mathrm{loc}}(U)$ if

$$\int_V |Du| < \infty$$

for every relatively compact open set $V \Subset U$.

The proof of the following theorem is an application of the Riesz representation theorem.

Theorem 4.58. *For every $u \in \mathrm{BV}_{\mathrm{loc}}(U)$ there exists a Radon measure μ on U and a μ-measurable mapping $\sigma \colon U \to \mathbb{R}^n$ such that*

(i) $|\sigma(x)| = 1$ *for μ-a.e. $x \in U$;*

(ii)

$$\int_U u \operatorname{div} g \, dx = -\int_U g \cdot \sigma \, d\mu$$

for every $g \in C^1_0(U; \mathbb{R}^n)$.

Remark 4.59. 1. If $u \in \mathrm{BV}_{\mathrm{loc}}(U)$, we denote by $\|Du\|$ the Radon measure μ given by Theorem 4.58 and by

$$[Du] = \|Du\| \llcorner \sigma$$

the vector valued measure $d[Du] = \sigma\, d\|Du\|$. Hence

$$\int_U u \operatorname{div} g = -\int_U g \cdot \sigma\, d\|Du\| = -\int_U g \cdot d[Du]$$

for $g \in C_0^1(U;\mathbb{R}^n)$.

2. If $u \in \mathrm{BV}(U)$ and $V \Subset U$ is an open subset, then

$$\|Du\|(V) = \sup\left\{\int_V u \operatorname{div} g\, dx \colon g \in C_0^1(U;\mathbb{R}^n), |g| \le 1\right\}.$$

Hence, using our earlier notation,

$$\int_V |Du| = \|Du\|(V).$$

Theorem 4.60 (Lower semicontinuity)**.** *Let $U \subset \mathbb{R}^n$ be open and $u_j \in \mathrm{BV}(U)$, $j \in \mathbb{N}$ such that $u_j \to u$ in $L^1_{\mathrm{loc}}(U)$. Then*

$$\int_U |Du| \le \liminf_{j\to\infty} \int_U |Du_j|. \tag{4.61}$$

Theorem 4.62. *The vector space $\mathrm{BV}(U)$ equipped with the BV-norm*

$$\|u\|_{\mathrm{BV}} := \|u\|_{L^1(U)} + \int_U |Du|$$

is a Banach space.

Functions in Sobolev spaces $W^{1,p}(U)$, $1 \le p < \infty$, can be approximated by $C^\infty(U)$ functions in the Sobolev norm

$$\|u\|_{1,p} := \|u\|_p + \||\nabla u|\|_p.$$

In fact, $W^{1,p}(U)$ is the completion of $C^\infty(U)$ in the Sobolev norm and since $\mathrm{BV}(U) \ne W^{1,1}(U)$, functions in $\mathrm{BV}(U)$ can not be approximated in the BV-norm. However,

Theorem 4.63 (Approximation)**.** *Let $u \in \mathrm{BV}(U)$. Then there exists a sequence $u_j \in C^\infty(U)$, $j \in \mathbb{N}$, such that*

$$\lim_{j\to\infty} \int_U |u_j - u| = 0,$$
$$\lim_{j\to\infty} \int_U |\nabla u_j| = \int_U |Du|.$$

Suppose that $u \in \mathrm{BV}(U)$ and $u_j \in C^\infty(U)$ are as above. For each $j \in \mathbb{N}$ let μ_j be the vector-valued Radon-measure defined by

$$\mu_j(B) = \int_{B\cap U} \nabla u_j\, dx$$

for Borel sets $B \subset \mathbb{R}^n$. Furthermore, let μ be the vector-valued Radon measure

$$\mu(B) = \int_{B\cap U} d[Du] = \int_{B\cap U} \sigma\, d\|Du\|.$$

Then $\mu_j \rightharpoonup \mu$.

Theorem 4.64 (Compactness)**.** *Let $U \subset \mathbb{R}^n$ be an open set. If $u_j \in \mathrm{BV}_{\mathrm{loc}}(U)$, $j \in \mathbb{N}$, is a sequence such that*

$$\sup_j \left(\|u_j\|_{L^1(W)} + \int_W |Du_j| \right) < \infty$$

for every $W \Subset U$, there exist a subsequence (u_{j_k}) and $u \in \mathrm{BV}_{\mathrm{loc}}(U)$ such that $u_{j_k} \to u$ in $L^1_{\mathrm{loc}}(U)$ and

$$\int_W |Du| \le \liminf_{j_k \to \infty} \int_W |Du_{j_k}|$$

for every $W \Subset U$.

Let us now return to consider n-currents. In the next theorem, which is a generalization of the Constancy theorem, we weaken the assumption $\partial T = 0$ to $\mathbf{M}(\partial T) < \infty$.

Theorem 4.65. *Let $T \in \mathcal{D}_n(U)$ such that $\mathbf{M}(\partial T) < \infty$ and $\mathbf{M}_W(T) < \infty$ for every $W \Subset U$. Then there exists $g \in \mathrm{BV}_{\mathrm{loc}}(U)$ such that*

$$T(\omega) = \int \varphi g \, dm_n, \tag{4.66}$$

for all $\omega = \varphi dx^1 \wedge \cdots \wedge dx^n \in \mathcal{D}^n(U)$.

The proof is a modification of the proof of the Constancy theorem. Instead of equality (4.52) we now have an estimate

$$\left| \int \frac{\partial \varphi}{\partial x_j} g_\varepsilon dm_n \right| = |T_\varepsilon(d\omega)| \le \sup |\eta_\varepsilon * \varphi| \mathbf{M}(\partial T) \le c_\varepsilon \mathbf{M}(\partial T) \tag{4.67}$$

if $\omega = \varphi dx^1 \wedge \cdots dx^{j-1} \wedge dx^{j+1} \wedge \cdots \wedge dx^n$, with $\varphi \in C^1_0(W)$, $|\varphi| \le 1$. Here c_ε is a constant that depends on ε and $c_\varepsilon \to 1$ as $\varepsilon \to 0$ since $\eta_\varepsilon * \varphi \to \varphi$ uniformly. We apply (4.67) with

$$\omega = (-1)^j \Phi_j dx^1 \wedge \cdots \wedge dx^{j-1} \wedge dx^{j+1} \wedge \cdots dx^n,$$

where $\Phi_j \in C^1_0(U)$, $\operatorname{supp} \Phi_j \subset W$, is the jth-coordinate function of $\Phi = (\Phi_1, \ldots, \Phi_n) \in C^1_0(U, \mathbb{R}^n)$, with $|\Phi| \le 1$. We obtain

$$\left| \int_U (\operatorname{div} \Phi) g_\varepsilon dm_n \right| = \left| \int_U \sum_{j=1}^n \frac{\partial \Phi_j}{\partial x_j} g_\varepsilon dm_n \right| \le n c_\varepsilon \mathbf{M}(\partial T) \le 2n \mathbf{M}(\partial T)$$

for all $\Phi \in C^1_0(U, \mathbb{R}^n)$, with $|\Phi| \le 1$, and $0 < \varepsilon < \operatorname{dist}(W, \partial U)$ small enough. Hence $g_\varepsilon \in \mathrm{BV}(W)$. It follows from the Poincaré's inequality for BV-functions (see e.g. [EG, 5.6.1], [Si, Lemma 6.4]) that g_ε is locally uniformly bounded in $L^1(U)$. We conclude (using Theorem 4.64) that there exists a sequence $\varepsilon_k \searrow 0$ such that $g_{\varepsilon_k} \to g$ in $L^1_{\mathrm{loc}}(U)$ with $g \in \mathrm{BV}_{\mathrm{loc}}(U)$. Moreover, it follows from (4.51) that

$$T(\omega) = \int_U \varphi g \, dm_n$$

for $\omega = \varphi dx^1 \wedge \cdots \wedge dx^n \in \mathcal{D}^n(U)$.

Writing an arbitrary $\alpha \in \mathcal{D}^{n-1}(U)$ as

$$\alpha = \sum_{j=1}^n (-1)^j \Phi_j dx^1 \wedge \cdots \wedge dx^{j-1} \wedge dx^{j+1} \wedge \cdots dx^n$$

we first observe that

$$d\alpha = (\operatorname{div}\Phi)dx^1 \wedge \cdots \wedge dx^n,\ \Phi = (\Phi_1, \dots, \Phi_n),$$

and therefore

$$\partial T(\alpha) = T(d\alpha) = T\big((\operatorname{div}\Phi)dx^1 \wedge \cdots \wedge dx^n\big) = \int_U (\operatorname{div}\Phi)g\, dm_n$$

by (4.66). Finally, it follows directly from definitions that

$$\mathbf{M}_W(T) = \int_W |g| dm_n$$

and

$$\mathbf{M}_W(\partial T) = \int_W |Dg| = \|Dg\|(W)$$

for every $W \Subset U$.

The last theorem in this subsection deals with restrictions of m-currents to subsets of $\mathbb{R}^n$ with "small" orthogonal projections onto $\mathbb{R}^m$. To state the result, we define for each multi-index $I = (i_1, \dots, i_m) \in \bigwedge(n, m)$ the orthogonal projection $P_I\colon \mathbb{R}^n \to \mathbb{R}^m$ by

$$P_I(x) = P_\alpha(x_1, \dots, x_n) = (x_{i_1}, \dots, x_{i_m}) \in \mathbb{R}^m.$$

Theorem 4.68. *Suppose that $E \subset \mathbb{R}^n$ is a closed subset of an open set $U \subset \mathbb{R}^n$ such that $\mathcal{H}^m\big(P_I E\big) = 0$ for every $I \in \bigwedge(n, m)$. Then $T \llcorner E = 0$ for all $T \in \mathcal{D}_m(U)$, with $\mathbf{M}_W(T) < \infty$ and $\mathbf{M}_W(\partial T) < \infty$.*

Proof. Let $\omega \in \mathcal{D}^m(U)$. We can write

$$\omega = \sum_{I \in \bigwedge(n,m)} \omega_I dx^I, \quad dx^I = dx^{i_1} \wedge \cdots \wedge dx^{i_m},\ I = (i_1, \dots, i_m).$$

Hence

$$\begin{aligned} T(\omega) &= \sum_I T(\omega_I dx^I) = \sum_I (T \llcorner \omega_I)(dx^I) \\ &= \sum_I (T \llcorner \omega_I)\big(P_I^*(dy^1 \wedge \cdots \wedge dy^m)\big) \qquad (4.69) \\ &= \sum_I P_{I\sharp}(T \llcorner \omega_I)(dy^1 \wedge \cdots \wedge dy^m), \end{aligned}$$

where we have denoted by $dy^1 \wedge \cdots \wedge dy^m$ the standard basis m-form in $\mathbb{R}^m$. The push-forward makes sense since $\operatorname{supp}(T \llcorner \omega_I)$ is a subset of $\operatorname{supp} \omega_I$ which is a compact subset of U.

For any $\beta \in \mathcal{D}^{m-1}(U)$

$$\begin{aligned} \partial(T \llcorner \omega_I)(\beta) &= (T \llcorner \omega_I)(d\beta) = T(\omega_I \beta) = T\big(d(\omega_I \beta)\big) - T(d\omega_I \wedge \beta) \\ &= \partial T(\omega_I \beta) - T(d\omega_I \wedge \beta), \end{aligned}$$

and so

$$\mathbf{M}_W\big(\partial(T \llcorner \omega_I)\big) \le \mathbf{M}_W(\partial T)|\omega_I| + \mathbf{M}_W(T)|d\omega_I|. \qquad (4.70)$$

We obtain

$$\mathbf{M}\big(\partial P_{I\sharp}(T\llcorner\omega_I)\big) = \mathbf{M}\big(P_{I\sharp}\partial(T\llcorner\omega_I)\big) \le c(n,m)\mathbf{M}\big(\partial(T\llcorner\omega_I)\big) < \infty$$

by (4.45), (4.70), and the assumptions $\mathbf{M}_W(T), \mathbf{M}_W(\partial T) < \infty\ \forall W \Subset U$. Therefore, by Theorem 4.65, there exists $g \in \mathrm{BV}(P_I U)$ such that

$$P_{I\sharp}(T\llcorner\omega_I)(\beta) = \int_{P_I U} \langle \beta, e_1 \wedge \cdots \wedge e_m\rangle g\, dm_m,$$

and hence

$$P_{I\sharp}(T\llcorner\omega_I)\llcorner P_I E = 0$$

since $m_m(P_I E) = 0$. Assuming, without loss of generality, that E is compact, we have

$$P_{I\sharp}(T\llcorner\omega_I) = P_{I\sharp}(T\llcorner\omega_I)\llcorner(\mathbb{R}^m \setminus P_I E) = P_{I\sharp}\Big((T\llcorner\omega_I)\llcorner\big(\mathbb{R}^n \setminus P_I^{-1}(P_I E)\big)\Big).$$

This implies

$$\begin{aligned} \mathbf{M}\big(P_{I\sharp}(T\llcorner\omega_I)\big) &\le \mathbf{M}\big((T\llcorner\omega_I)\llcorner\big(\mathbb{R}^n \setminus P_I^{-1}(P_I E)\big)\big) \\ &\le \mathbf{M}\big((T\llcorner\omega_I)\llcorner(\mathbb{R}^n \setminus E)\big) \\ &\le \mathbf{M}_W\big(T\llcorner(\mathbb{R}^n \setminus E)\big)\big)|\omega_I| \end{aligned} \tag{4.71}$$

for every open W such that $\operatorname{supp}\omega \subset W \Subset U$. Combining (4.69) and (4.71) we get

$$\mathbf{M}_W(T) \le c\mathbf{M}_W\big(T\llcorner(\mathbb{R}^n \setminus E)\big)$$

for all open $W \Subset U$. In particular,

$$\mathbf{M}(T\llcorner E) = \mathbf{M}_W(T\llcorner E) \le c\mathbf{M}_W\big(T\llcorner(\mathbb{R}^n \setminus E)\big)$$

for all $W \Subset U$, with $E \subset W$. Choosing a descending sequence of open sets $W_i \Subset U$ such that $E = \cap_i W_i$, we get

$$\mathbf{M}(T\llcorner E) \le c\mathbf{M}_{W_i}\big(T\llcorner(\mathbb{R}^n \setminus E)\big) \to 0$$

which implies $T\llcorner E = 0$. □

4.72 Rectifiable currents

Definition 4.73. An m-current $T \in \mathcal{D}_m(U)$ in an open set $U \subset \mathbb{R}^n$ is called a *rectifiable m-current* if there exist

1. an m-rectifiable Borel set $E \subset U$, with $\mathcal{H}^m(E) < \infty$;
2. an $\mathcal{H}^m$-integrable positive function $\theta\colon E \to (0,\infty)$, and
3. an $\mathcal{H}^m$-measurable mapping $\vec{T}\colon E \to \bigwedge_m(\mathbb{R}^n)$ such that, for cH^m-a.e. $x \in E$, $\vec{T}(x) = v_1(x)\wedge\cdots\wedge v_m(x)$ where $v_1(x),\ldots,v_m(x)$ is an orthonormal basis of the approximate tangent space $T_x^m E$, and that

$$T(\omega) = \int_E \big\langle \omega(x), \vec{T}(x)\big\rangle \theta(x) d\mathcal{H}^m(x)$$

for all $\omega \in \mathcal{D}^m(U)$. Note that $|\vec{T}(x)| = 1$ for $\mathcal{H}^m$-a.e. $x \in E$.

The function θ is called the *multiplicity* of T and $\vec{T}$ is called the *orientation* for T. We write $T = \tau(E, \theta, \vec{T})$. Such a current T is called an *integer multiplicity (rectifiable) m-current*, denoted $T \in \mathcal{R}_m(U)$, if θ is integer valued.

Example 4.74. (1) If $T_1, T_2 \in \mathcal{R}_m(U)$ and $p_1, p_2 \in \mathbb{N}$, then $p_1T_1 + p_2T_2 \in \mathcal{R}_m(U)$.

(2) If $T_1 = \tau(E_1, \theta_1, \vec{T_1}) \in \mathcal{R}_m(U)$ and $T_2 = \tau(E_2, \theta_2, \vec{T_2}) \in \mathcal{R}_k(V)$, then

$$T_1 \times T_2 = \tau(E_1 \times E_2, \theta_1\theta_2, \vec{T_1} \wedge \vec{T_2}) \in \mathcal{R}_{m+k}(U \times V).$$

(3) If $f\colon U \to V$ is Lipschitz, $T = \tau(E, \theta, \vec{T}) \in \mathcal{R}_m(U)$, and $f|\operatorname{supp} T$ is proper, we can define $f_\sharp T \in \mathcal{D}_m(V)$ by

$$f_\sharp T(\omega) = \int_E \langle \omega\big(f(x)\big), \bigwedge{}^m d^E f_x \vec{T}(x)\rangle \theta\big(f(x)\big) d\mathcal{H}^m(x)$$

for $\omega \in \mathcal{D}^m(V)$. Since

$$\left|\bigwedge{}^m d^E f_x \vec{T}(x)\right| = J_f^E(x),$$

we get from the area formula that

$$f_\sharp T(\omega) = \int_{fE} \left\langle \omega(y), \sum_{x \in f^{-1}(y) \cap E_+} \theta(x) \frac{\bigwedge^m d^E f_x \vec{T}(x)}{|\bigwedge^m d^E f_x \vec{T}(x)|} \right\rangle d\mathcal{H}^m(y), \tag{4.75}$$

where $E_+ = \{x \in E\colon J_f^E(x) > 0\}$. Notice that fE is m-rectifiable, and therefore the approximate tangent space $T_y^m fE$ exists at $\mathcal{H}^m$-a.e. $x \in fE$. Hence at points $y \in fE$ where $T_y^m fE$ exists and for which $T_x^m E$ and $d^E f_x$ exist for all $x \in f^{-1}(y) \cap E_+$, we have

$$\frac{\bigwedge^m d^E f_x \vec{T}(x)}{|\bigwedge^m d^E f_x \vec{T}(x)|} = \pm \tau_1 \wedge \cdots \wedge \tau_m,$$

where $\tau_1, \ldots, \tau_m$ is an orthonormal basis of $T_y^m fE$. Hence we obtain from (4.75)

$$f_\sharp T(\omega) = \int_{fE} \langle \omega(y), \vec{S}(y) \rangle N(y) d\mathcal{H}^m(y),$$

where $\vec{S}(y)$ is an orientation of $T_y^m fE$ and $N(y)$ is a positive integer satisfying

$$\sum_{x \in f^{-1}(y) \cap E_+} \theta(x) \frac{\bigwedge^m d^E f_x \vec{T}(x)}{|\bigwedge^m d^E f_x \vec{T}(x)|} = N(y)\vec{S}(y).$$

In conclusion, $f_\sharp T \in \mathcal{R}_m(V)$.

Definition 4.76. An m-current $P \in \mathcal{D}_m(U)$ is a *polyhedral (m-)chain* if there exist m-dimensional oriented simplices $\pi_1, \ldots, \pi_k \subset U$ and $p_1, \ldots, p_k \in \mathbb{R}$ such that

$$P = \sum_{i=1}^k p_i[\pi_i].$$

If $p_1, \ldots, p_k \in \mathbb{Z}$, P is called an *integral polyhedral chain* and we denote

$$P \in \mathcal{P}_m(U).$$

Recall that an m-simplex π is the convex hull of its $m+1$ affinely independent vertices $a_0, \dots, a_m \in \mathbb{R}^m$, that is $a_1 - a_0, a_2 - a_0, \dots, a_m - a_0$ are linearly independent and

$$\pi = \left\{ \sum_{i=0}^{m} \lambda_i a_i \colon \sum_{i=0}^{m} \lambda_i = 1, \lambda_i \geq 0 \ \forall i \right\}.$$

Theorem 4.77. *If $T_i \in \mathcal{R}_m(U)$ is a sequence (of integer multiplicity rectifiable m-currents) with*

$$\sup_{i \in \mathbb{N}} \left(\mathbf{M}_W(T_i) + \mathbf{M}_W(\partial T_i) \right) < \infty$$

for all $W \Subset U$, then there exist a subsequence T_{i_j} and $T \in \mathcal{R}_m(U)$ such that $T_{i_j} \to T$.

Note that the existence of a subsequence and an m-current $T \in \mathcal{D}_m(U)$ such that $T_{i_j} \to T$ follows from the Banach-Alaoglu theorem; see Theorem 4.36. The difficulty is to prove that T is an integer multiplicity rectifiable current; we will return to this later.

The next theorem gives a criterion of rectifiability.

Theorem 4.78. *Let $T \in \mathcal{D}_m(U)$ with $\mathbf{M}(T) < \infty$. Then $T \in \mathcal{R}_m(U)$ if and only if for every $\varepsilon > 0$ there exist $P \in \mathcal{P}_m(\mathbb{R}^d)$, $d \geq m$, and a Lipschitz map $f \colon \mathbb{R}^d \to \mathbb{R}^n$ such that*

$$\mathbf{M}(T - f_\sharp P) < \varepsilon. \tag{4.79}$$

Proof. Idea: $\boxed{\Leftarrow}$ Let $T \in \mathcal{D}_m(U)$ with $\mathbf{M}(T) < \infty$. Each m-simplex is a subset of $\mathbb{R}^m \subset \mathbb{R}^d$ and hence $f_\sharp P$ is an m-rectifiable integer multiplicity current. Apply (4.79) with $\varepsilon_i \searrow 0$, i.e. let $P_i \in \mathcal{P}_m(\mathbb{R}^d)$ such that

$$\mathbf{M}(T - f_\sharp P_i) < \varepsilon_i.$$

Then, for every $\omega \in \mathcal{D}^m(U)$,

$$|T(\omega) - f_\sharp P_i(\omega)| = |(T - f_\sharp P_i)(\omega)| \leq \mathbf{M}(T - f_\sharp P_i)|\omega| \to 0.$$

Hence T, as a limit of integer multiplicity m-currents $f_\sharp P_i \in \mathcal{R}_m(\mathbb{R}^n)$, is an integer multiplicity m-current; see Lemma 4.80.

$\boxed{\Rightarrow}$ Let $\varepsilon > 0$ and $T = \tau(E, \theta, \vec{T}) \in \mathcal{R}_m(U)$. We may assume (ignoring a set of $\mathcal{H}^m$-measure zero) that E is a countable union of Lipschitz images $f_i A_i)$ of subsets $A_i \subset R^m$. Furthermore, we may assume that the sets A_i are disjoint and that $\theta | f_i A_i$ takes a constant value $\theta_i \in \mathbb{N}$. Then we take θ_i copies $A_{i,j}$, $j = 1, \dots, \theta_i$ of A_i such that all the sets $A_{i,j}$, $i \in \mathbb{N}$, $j = 1, \dots, \theta_i$, are disjoint. Now we can define a Lipschitz map (after applying the corollary of the McShane-Whitney extension theorem) $f \colon \mathbb{R}^m \to \mathbb{R}^n$ such that $f A_{i,j} = f A_i$ and that f preserves orientation. On the other hand, each $A_{i,j}$ can be approximated by finitely many m-simplices and hence T can be approximated (in mass) by an integral polyhedral chain. □

In the above proof, the step $\mathbf{M}(T - f_\sharp P_i) \to 0 \Rightarrow f_\sharp P_i \to T$ is relatively easy.

Lemma 4.80. *The set of integer multiplicity rectifiable currents in $\mathcal{D}_m(U)$ is complete with respect to the family of seminorms $\{\mathbf{M}_W \colon W \Subset U\}$.*

Proof. Let $T_i = \tau(E_i, \theta_i, \vec{T}_i) \in \mathcal{R}_m(U)$, $i \in \mathbb{N}$ be a Cauchy sequence with respect to the family $\{\mathbf{M}_W \colon W \Subset U\}$. Then

$$\mathbf{M}_W(T_i - T_j) = \int_W |\theta_i \vec{T}_i - \theta_j \vec{T}_j| \, d\mathcal{H}^m < \varepsilon(W, j) \tag{4.81}$$

for $i \geq j$, where $\varepsilon(W,j) \searrow 0$ as $j \to \infty$, and we have made a convention $\theta_k = 0$ and $\vec{T}_k = 0$ in $U \setminus E_k$. Since $|\vec{T}_k(x)| \equiv 1$ in E_k, we get

$$\int_W |\theta_i - \theta_j| d\mathcal{H}^m < \varepsilon(W,j), \quad i \geq j. \tag{4.82}$$

Hence $\theta_i \to \theta$ in $L^1_{\mathrm{loc}}(U, \mathcal{H}^m)$, where θ is integer valued. From (4.82), we get

$$\mathcal{H}^m\big((E_+ \setminus E_j) \cup (E_j \setminus E_+)\big) < \varepsilon(W,j), \tag{4.83}$$

where $E_+ = \{x \in U \colon \theta(x) > 0\}$. Since

$$\theta_i|\vec{T}_i - \vec{T}_j| = |\theta_i\vec{T}_i - \theta_j\vec{T}_j + (\theta_j - \theta_i)\vec{T}_j| \leq |\theta_i\vec{T}_i - \theta_j\vec{T}_j| + |\theta_j - \theta_i||\vec{T}_j|,$$

we have

$$\int_W \theta_i|\vec{T}_i - \vec{T}_j| d\mathcal{H}^m < 2\varepsilon(W,j), \quad i \geq j,$$

and therefore $\vec{T}_i$ converges in $L^1_{\mathrm{loc}}(\mathcal{H}^m)$ to $\vec{T} \colon U \to \bigwedge_m(\mathbb{R}^n)$, where $\vec{T}$ is simple and $|\vec{T}| = 1$ in E_+. Since $\vec{T}_j \in \bigwedge_m(T_xE_j)$ for $\mathcal{H}^m$-a.e. $x \in E_j$ and $T_xE_j = T_xE_+$ in $E_j \cap E_+$ except a set of $\mathcal{H}^m$-measure $\leq \varepsilon(W,j)$ by (4.83), we conclude that $\vec{T} \in \bigwedge_m(T_xE_+)$, and so $\mathbf{M}(T - T_j) \to 0$, with $T = \tau(E_+, \theta, \vec{T}) \in \mathcal{R}_m(U)$. □

4.84 Slicing

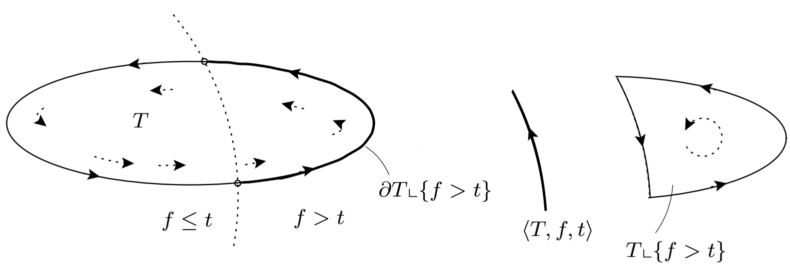
In this subsection we introduce the slicing of a current by level sets of a Lipschitz function. [Recall the co-area formula and, in particular, Theorem 2.59, where we "sliced" an m-rectifiable set E by level sets of a Lipschitz function.]

Definition 4.85. A current $T \in \mathcal{D}_m(U)$ is *normal*, denoted by $T \in \mathcal{N}_m(U)$, if $\operatorname{supp} T$ is compact and

$$\mathbf{M}(T) + \mathbf{M}(\partial T) < \infty.$$

Definition 4.86. Let $T \in \mathcal{N}_m(U)$ be normal and $f \colon \mathbb{R}^n \to \mathbb{R}$ a Lipschitz map. The *slice* of T with f and $t \in \mathbb{R}$ is

$$\langle T, f, t\rangle := (\partial T)\llcorner\{x \colon f(x) > t\} - \partial\big(T\llcorner\{x \colon f(x) > t\}\big) \in \mathcal{D}_{m-1}(U).$$



Theorem 4.87. *The slices have the properties:*

(1)

$$\langle T, f, t\rangle = \partial\big(T\llcorner\{x \colon f(x) < t\}\big) - (\partial T)\llcorner\{x \colon f(x) < t\}$$

except at most countably many t;

(2)
$$\operatorname{supp}\langle T, f, t\rangle \subset f^{-1}(t) \cap \operatorname{supp} T;$$

(3)
$$\mathbf{M}\big(\langle T, f, t\rangle\big) \leq \operatorname{Lip}(f) \liminf_{h \searrow 0} \frac{1}{h} \mu_T\big(\{x \colon t \leq f(x) \leq t+h\}\big);$$

(4)
$$\int_a^b \mathbf{M}\big(\langle T, f, t\rangle\big) dt \leq \operatorname{Lip}(f) \mu_T\big(\{x \colon a < f(x) < b\}\big);$$

(5)
$$\partial\langle T, f, t\rangle = -\langle \partial T, f, t\rangle;$$

(6) $\langle T, f, t\rangle$ is normal for almost every t.

Proof. Idea of (some) proofs: (1) holds for every t for which
$$(\mu_T + \mu_{\partial T})\big(\{x \colon f(t) = t\}\big) = 0.$$
(2) is easy. To prove (3), we approximate the characteristic function
$$\chi_{\{x \colon f(x) > t\}}$$
by a sequence of C^∞ functions g such that $g(x) = 0$ if $f(x) \leq t,\ f(x) = 1$ if $f(x) \geq t+h$, and
$$\operatorname{Lip}(g) \leq \frac{\lambda \operatorname{Lip}(f)}{h},$$
where $\lambda > 1,\ \lambda \approx 1$. Then
$$\begin{aligned} \mathbf{M}\big(\langle T, f, t\rangle\big) &\approx \mathbf{M}\big((\partial T)\llcorner g - \partial(T \llcorner g)\big) \\ &= M(T \llcorner dg) \\ &\leq \operatorname{Lip}(g) \mu_T\big(\{x \colon t \leq f(x) \leq t+h\}\big). \end{aligned}$$
(4) follows from (3) by integration, (5) is clear, and finally (6) follows from (4) and (5). □

Next we slice integer multiplicity rectifiable currents.

Theorem 4.88. *Let $T = \tau(E, \theta, \vec{T}) \in \mathcal{R}_m(U)$, with $\mathbf{M}(\partial T) < \infty$ and let $f \colon \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz function. Then for a.e. $t \in \mathbb{R}$:*

(1) $\langle T, f, t\rangle = \tau(E_t, \theta_t, \vec{T_t})$, where
$$\begin{aligned} E_t &= E \cap f^{-1}(t), \\ \theta_t(x) &= \begin{cases} \theta(x), & \text{if } x \in E_t \text{ and } \nabla^E f(x) \neq 0; \\ 0, & \text{if } x \in E_t \text{ and } \nabla^E f(x) = 0, \end{cases} \\ \vec{T_t}(x) &= \vec{T}(x) \llcorner \frac{\nabla^E f(x)}{|\nabla^E f(x)|}, \end{aligned}$$

(2)
$$\int_{-\infty}^{\infty} \mathbf{M}\big(\langle T, f, t\rangle\big) dt = \int_E |\nabla^E f| |\theta| \, d\mathcal{H}^m \leq \operatorname{Lip}(f) \mathbf{M}(T),$$

(3) $\langle T, f, t\rangle \in \mathcal{R}_{m-1}(U)$*, with* $\mathbf{M}\big(\partial\langle T, f, t\rangle\big) < \infty$ *and* $\partial\langle T, f, t\rangle = -\langle \partial T, f, t\rangle$*.*

The *interior multiplication* $\llcorner\colon \bigwedge_q(V) \times \bigwedge^p(V) \to \bigwedge_{q-p}(V)$ is characterized by the condition

$$\langle v \llcorner \alpha, \beta\rangle = \langle v, \alpha \wedge \beta\rangle$$

whenever $v \in \bigwedge_q(V)$, $\alpha \in \bigwedge^p(V)$, $\beta \in \bigwedge^{q-p}(V)$. Moreover, there is the standard biduality between finite dimensional inner product spaces $\bigwedge_m(V)$ and $\bigwedge^m(V)$. That is, for every $\eta \in \bigwedge^m(V)$ there exists a unique $w \in \bigwedge_m(V)$ such that

$$\langle v, w\rangle = \langle \eta, v\rangle \quad \forall v \in \bigwedge{}_m(V). \tag{4.89}$$

Hence, in particular, $\vec{T}(x) \llcorner \nabla^E f(x) \in \bigwedge_{m-1}(T_x^{m-1} E_t)$ is characterized by the property

$$\langle \vec{T}(x) \llcorner \nabla^E f(x), \eta\rangle = \langle \vec{T}(x), d^E f_x \wedge \eta\rangle$$

for all $\eta \in \bigwedge^{m-1}(T_x^{m-1} E_t)$.

Proof of Theorem 4.88. Let $h\colon \mathbb{R}^n \to \mathbb{R}$ be Lipschitz and let $h_\varepsilon = \eta_\varepsilon * h$ be as before. Then $h_\varepsilon \to h$ locally uniformly as $\varepsilon \to 0$. Now

$$\partial T(h_\varepsilon \omega) = T\big(d(h_\varepsilon \omega)\big) = T(dh_\varepsilon \wedge \omega) + T(h_\varepsilon d\omega)$$

for all $\omega \in \mathcal{D}^m(U)$. Here

$$\partial T(h_\varepsilon \omega) = \int \langle \overrightarrow{\partial T}, h_\varepsilon \omega\rangle d\mu_{\partial T} \to \int \langle \overrightarrow{\partial T}, h\omega\rangle d\mu_{\partial T} = (\partial T \llcorner h)(\omega)$$

and

$$T(h_\varepsilon d\omega) = \int \langle \vec{T}, h_\varepsilon d\omega\rangle d\mu_T \to \int \langle \vec{T}, h d\omega\rangle d\mu_T = (T \llcorner h)(d\omega) = \partial(T \llcorner h)(\omega)$$

as $\varepsilon \to 0$. So,

$$(\partial T \llcorner h)(\omega) = \lim_{\varepsilon \to 0} T(dh_\varepsilon \wedge h) + \partial(T \llcorner h)(\omega),$$

where

$$\begin{aligned} T(dh_\varepsilon \wedge h) &= \int_E \langle \vec{T}(x), dh_\varepsilon(x) \wedge \omega\rangle \theta(x) d\mathcal{H}^m(x) \\ &= \int_E \langle \vec{T}(x), d^E h_\varepsilon(x) \wedge \omega\rangle \theta(x) d\mathcal{H}^m(x) \\ &\overset{(4.89)}{=} \int_E \langle \vec{T}(x) \llcorner \nabla^E h_\varepsilon(x), \omega(x)\rangle \theta(x) d\mathcal{H}^m(x) \\ &\to \int_E \langle \vec{T}(x) \llcorner \nabla^E h(x), \omega(x)\rangle \theta(x) d\mathcal{H}^m(x). \end{aligned}$$

[2] Hence we get from the convergences above that

$$(\partial T \llcorner h)(\omega) = \int_E \langle \vec{T}(x) \llcorner \nabla^E h(x), \omega(x)\rangle \theta(x) d\mathcal{H}^m(x) + \partial(T \llcorner h)(\omega)$$

[2]The last convergence holds since $\nabla^E h_\varepsilon \to \nabla^E h$ weakly in $L^2(\mathcal{H}^m \llcorner \theta)$ which, in turn, can be proven by noticing that $E = \sqcup_{i=0}^\infty E_i$, $\mathcal{H}^m(E_0) = 0$, and $E_i \subset M_i$, with M_i a C^1-smooth submanifold.

for Lipschitz functions $h\colon \mathbb{R}^n \to \mathbb{R}$.

Let then $f\colon \mathbb{R}^n \to \mathbb{R}$ be Lipschitz, $t \in \mathbb{R}$, and $\varepsilon > 0$. Define a continuous function $\gamma_\varepsilon\colon \mathbb{R} \to \mathbb{R}$ by

$$\gamma_\varepsilon(s) = \begin{cases} 0, \text{ if } s < t-\varepsilon, \\ \text{linear, if } t-\varepsilon \le s \le t, \\ 1, \text{ if } s > t \end{cases}$$

and $g_\varepsilon = \gamma_\varepsilon \circ f$. We then have

$$(\partial T \llcorner g_\varepsilon)(\omega) = \int_E \langle \vec{T}(x) \llcorner \nabla^E g_\varepsilon(x), \omega(x)\rangle \theta(x) d\mathcal{H}^m(x) + \partial(T\llcorner g_\varepsilon)(\omega). \tag{4.90}$$

Now

$$g_\varepsilon(x) \xrightarrow{\varepsilon\to 0} \chi_{\{f>t\}}(x),$$

so

$$(\partial T \llcorner g_\varepsilon)(\omega) = \int \langle \overrightarrow{\partial T}, g_\varepsilon \omega\rangle d\mu_{\partial T} \to \int \langle \overrightarrow{\partial T}, \chi_{\{f>t\}}\omega\rangle d\mu_{\partial T} = \partial T \llcorner \{f>t\}(\omega). \tag{4.91}$$

Similarly,

$$(T \llcorner g_\varepsilon)(d\omega) = \int \langle \vec{T}, g_\varepsilon d\omega\rangle d\mu_T \to \int \langle \vec{T}, \chi_{\{f>t\}} d\omega\rangle d\mu_T = \partial\big(T\llcorner\{f>t\}\big)(\omega). \tag{4.92}$$

By the chain rule

$$\begin{aligned} \nabla^E g_\varepsilon(x) &= \nabla^E(\gamma_\varepsilon \circ f)(x) = \gamma'\big(f(x)\big)\nabla^E f(x) \\ &= \begin{cases} 0, \text{ if } f(x) < t-\varepsilon \text{ or } f(x) > t, \\ \frac{1}{\varepsilon}\nabla^E f(x), \text{ if } t-\varepsilon < f(x) < t, \end{cases} \end{aligned}$$

so

$$\begin{aligned} \int_E \langle \vec{T}(x)\llcorner\nabla^E g_\varepsilon(x), \omega(x)\rangle\theta(x) d\mathcal{H}^m(x) &= \frac{1}{\varepsilon} \int\limits_{\{x\in E\colon\, t-\varepsilon<f(x)<t\}} \langle \vec{T}(x)\llcorner\nabla^E f(x), \omega(x)\rangle \theta(x) d\mathcal{H}^m(x) \\ &= \frac{1}{\varepsilon} \int\limits_{\{x\in E\colon\, t-\varepsilon<f(x)<t\}} \left\langle \vec{T}\llcorner\frac{\nabla^E f}{|\nabla^E f|}, \omega \right\rangle |\nabla^E f| \theta d\mathcal{H}^m \\ &= \frac{1}{\varepsilon}\int_{t-\varepsilon}^t \left(\int_{E_s} \langle \vec{T}_s, \omega\rangle \theta_s \, d\mathcal{H}^{m-1}\right) ds \\ &\to \int_{E_t} \langle \vec{T}_t, \omega\rangle\theta_t \, d\mathcal{H}^{m-1} \end{aligned}$$

for a.e. $t \in \mathbb{R}$. Recalling (4.90)-(4.92) and the definition of $\langle T, f, t\rangle$ we get

$$\begin{aligned} \langle T, f, t\rangle &= (\partial T)\llcorner\{f>t\} - \partial(T\llcorner\{f>t\}) \\ &= \lim_{\varepsilon\to 0}(\partial T\llcorner g_\varepsilon) - \lim_{\varepsilon\to 0}\partial(T\llcorner g_\varepsilon) \\ &= \lim_{\varepsilon\to 0}\int_E \big\langle T\llcorner\nabla^E g_\varepsilon, \cdot\big\rangle \theta d\mathcal{H}^m \\ &= \int_{E_t} \langle \vec{T}_t, \cdot\rangle \theta_t d\mathcal{H}^{m-1} \\ &= \tau(E_t, \theta_t, \vec{T}_t), \end{aligned}$$

and therefore (1) holds. (2) follows from (1) and the co-area formula. (3) follows from (1), Theorem 2.59, and Theorem 4.87 (5). □

It is possible to slice an current $T \in \mathcal{D}_m(U)$ with a Lipschitz map $f = (f_1, \dots, f_k))\colon \mathbb{R}^n \to \mathbb{R}^k,\ k \le m$, and a value $y = (y_1, \dots, y_k) \in \mathbb{R}^k$ by iterating the slicing with f_i and y_i:

$$\langle T, f, y\rangle = \langle \langle \cdots \langle \langle T, f_1, y_1\rangle, f_2, y_2\rangle \cdots \rangle, f_k, y_k\rangle \in \mathcal{D}_{m-k}(U).$$

4.93 Deformation theory

The deformation theorem is one of the fundamental results and it provides a useful approximation of a normal current T by a polyhedral chain P lying on a certain m-skeleton such that the error is of the form $T - P = \partial R + S$. The main error term is ∂R, where R is the $(m+1)$-dimensional surface through which T is deformed to P. The other error term S arises in moving ∂T into the skeleton.

We will only state the result and sketch the (long and technical) proof. First we introduce some notation: Fix $k, m, n \in \mathbb{N},\ 0 < m < n$, and $\varepsilon > 0$. We denote by

$$Q_\varepsilon = [0, \varepsilon]^n \subset \mathbb{R}^n$$

the closed n-dimensional cube of side length ε and by

$$L_{\varepsilon,k} = \bigcup_{j=1}^{k} \mathcal{L}_{\varepsilon,j} = \{\pi\colon \pi\ j\text{-dimen. closed face of some } Q_\varepsilon + p\varepsilon,\ p \in \mathbb{Z}^n\}$$

the k-skeleton of mesh ε. Thus the elements of

- $\mathcal{L}_{\varepsilon,0}$ are singletons (vertices),
- $\mathcal{L}_{\varepsilon,1}$ are closed line segments (edges) of length ε,
- $\mathcal{L}_{\varepsilon,2}$ are closed squares of side length ε, $\cdots$
- $\mathcal{L}_{\varepsilon,n}$ are the closed n-cubes $Q_\varepsilon + p\varepsilon,\ p \in \mathbb{Z}^n$, of side length ε.

Moreover, we denote by $V_{\varepsilon,1}, \dots, V_{\varepsilon,N},\ N = \binom{n}{m+1}$ the $(m+1)$-dimensional affine subspaces of $\mathbb{R}^n$ that contain some $(m+1)$-face of Q_ε. Finally,

$$P_{\varepsilon,j}\colon \mathbb{R}^n \to V_{\varepsilon,j}$$

denotes the orthogonal projection onto $V_{\varepsilon,j}$.

Theorem 4.94 (Deformation theorem)**.** *Let $\varepsilon > 0$ and $T \in \mathcal{D}_m(\mathbb{R}^n)$, with $\mathbf{M}(T) + \mathbf{M}(\partial T) < \infty$. Then there are $P, S \in \mathcal{D}_m(\mathbb{R}^n)$ and $R \in \mathcal{D}_{m+1}(\mathbb{R}^n)$ such that*

$$T - P = \partial R + S,$$

where P, R, and S satisfy the following:

$$P = \sum_{\pi \in \mathcal{L}_{\varepsilon,m}} \alpha_\pi [\pi], \quad \alpha_\pi \in \mathbb{R}, \tag{4.95}$$

$$\mathbf{M}(P) \le c\mathbf{M}(T), \quad \mathbf{M}(\partial P) \le c\mathbf{M}(\partial T), \tag{4.96}$$

$$\mathbf{M}(R) \le c\varepsilon \mathbf{M}(T), \quad \mathbf{M}(S) \le c\varepsilon \mathbf{M}(\partial T), \tag{4.97}$$

with $c = c(n,m)$, *and*

$$\operatorname{supp} P \cup \operatorname{supp} R \subset \{x \in \mathbb{R}^n\colon\ \operatorname{dist}(x, \operatorname{supp} T) < 2\varepsilon\sqrt{n}\},$$
$$\operatorname{supp} \partial P \cup \operatorname{supp} S \subset \{x \in \mathbb{R}^n\colon\ \operatorname{dist}(x, \operatorname{supp} \partial T) < 2\varepsilon\sqrt{n}\}.$$

If $T \in \mathcal{R}_m(\mathbb{R}^n)$, *also* P *and* R *can be chosen to be integer multiplicity with* $\alpha_\pi \in \mathbb{Z}$. *If, in addition,* $\partial T \in \mathcal{R}_{m-1}(\mathbb{R}^n)$, *also* S *can be chosen to be integer multiplicity.*

For the proof, we may assume that $\varepsilon = 1$. Indeed, the "scaled version" 4.94 follows from the "unscaled" one where $\varepsilon = 1$ by first applying the homothety $x \mapsto x/\varepsilon$, then applying the "unscaled version" and then scaling back by $x \mapsto \varepsilon x$. In particular, the linear dependence of the constant $c\varepsilon$ in (4.97) on ε is then obvious.

The main tool in the proof of the deformation theorem is the following lemma that provides a suitable class of retractions to push-forward T into the m-skeleton $L_{1,m}$ (in the unscaled version). We denote by $q = (1/2, \dots, 1/2)$ the center of the unit cube $Q = Q_1$ and abbreviate $L_k = L_{1,k}$, $\mathcal{L}_k = \mathcal{L}_{1,k}$, and $P_j = P_{1,j}$. Given a point $a \in B(q, 1/4)$, we denote

$$L_{n-m-1}(a) = a + L_{n-m-1} \quad \text{(shifted skeleton)}$$

and

$$L_{n-m-1}(a;\rho) = \{x \in \mathbb{R}^n\colon\ \operatorname{dist}\big(x, L_{n-m-1}(a)\big) < \rho\},\ \ \rho \in (0, 1/4).$$

Then

$$\operatorname{dist}\big(L_{n-m-1}(a), L_m\big) \ge 1/4 \quad \forall a \in B(q, 1/4).$$

Lemma 4.98. *For every* $a \in B(q, 1/4)$ *there is a locally Lipschitz map*

$$\psi\colon \mathbb{R}^n \setminus L_{n-m-1}(a) \to \mathbb{R}^n \setminus L_{n-m-1}(a)$$

such that

$$\psi\big(Q \setminus L_{n-m-1}(a)\big) = Q \cap L_m,$$
$$\psi | Q \cap L_m = \operatorname{id}_{Q \cap L_m},$$
$$|Df(x)| \le c/\rho$$

for m_n*-a.e.* $x \in Q \setminus L_{n-m-1}(a;\rho)$, $\rho \in (0, 1/4)$, *with* $c = c(n,m)$ *and that*

$$\psi(z + x) = z + \psi(x)$$

for all $x \in \mathbb{R}^n \setminus L_{n-m-1}(a)$ *and* $z \in \mathbb{Z}^n$.

Thus ψ is a $\mathbb{Z}^n$-periodic retraction of $\mathbb{R}^n \setminus L_{n-m-1}(a)$ onto L_m. The *rough idea* is then to define

$$\tilde{P} = \psi_\sharp T,$$
$$R = h_\sharp\big([[0,1]] \times T\big),$$

and

$$S_1 = h_\sharp\big([[0,1]] \times \partial T\big),$$

where $h(t,x) = tx + (1-t)\psi(x)$, so that the homotopy formula gives

$$T = \tilde{P} + \partial R + S_1.$$

Choosing the point $a \in B(q, 1/4)$ properly (depending on T) we may get estimates

$$\mathbf{M}(\tilde{P}) \leq c\mathbf{M}(T),$$
$$\mathbf{M}(\partial\tilde{P}) \leq c\mathbf{M}(\partial T),$$
$$\mathbf{M}(R) \leq c\mathbf{M}(T),$$

and

$$\mathbf{M}(S_1) \leq c\mathbf{M}(\partial T).$$

We notice that $\tilde{P}$ need not be a polyhedral chain. It is used to choose appropriate multiplicities of the m-faces in the m-skeleton. For each m-face $F \in L_m$, $\tilde{P} \llcorner F$ corresponds by Theorem 4.65 to a BV-function θ_F so that

$$\mathbf{M}(\tilde{P} \llcorner F) = \int_F |\theta_F| d\mathcal{H}^m, \quad \mathbf{M}\big((\partial\tilde{P})\llcorner F\big) = \int_F |D\theta_F| d\mathcal{H}^m.$$

Letting then

$$m_F = \frac{1}{\mathcal{H}^m(F)} \int_F \theta_F d\mathcal{H}^m,$$

we define

$$P = \sum_{F \in L_m} m_F[F]$$

and

$$S = S_1 + (\tilde{P} - P).$$

In the proof of the mass estimates, for instance, slicing is used.

Next we give some applications of the deformation theorem.

Theorem 4.99 (Isoperimetric inequality). *If $T \in \mathcal{R}_m(\mathbb{R}^n)$ with $\operatorname{supp}(T)$ compact and $\partial T = 0$, there exists $R \in \mathcal{R}_{m+1}(\mathbb{R}^n)$, with $\operatorname{supp}(R)$ compact, $\partial R = T$ and*

$$\mathbf{M}(R)^{m/(m+1)} \leq C_{n,m}\mathbf{M}(T).$$

Proof. We may assume $T \neq 0$. Choose $\varepsilon > 0$ so that $\varepsilon^m = 2c\mathbf{M}(T)$, where $c = c(n,n)$ is the constant in the deformation theorem. By the deformation theorem, there are P, R, and S such that

$$T = P + \partial R + S,$$

where $R \in \mathcal{R}_{m+1}(\mathbb{R}^n)$, with compact support,

$$P = \sum_{\pi \in \mathcal{L}_{\varepsilon,m}} \alpha_\pi[\pi], \quad \alpha_\pi \in \mathbb{Z},$$
$$\mathbf{M}(P) \leq c\mathbf{M}(T),$$
$$\mathbf{M}(S) \leq c\varepsilon\mathbf{M}(\partial T),$$

and

$$\mathbf{M}(R) \leq c\varepsilon\mathbf{M}(T) = c(2c)^{1/m}\mathbf{M}(T)^{(m+1)/m}.$$

Since $\partial T = 0$, we obtain from above that $S = 0$. On the other hand,

$$\mathbf{M}(P) = \sum_{\pi \in \mathcal{L}_{\varepsilon,m}} |\alpha_\pi| \mathcal{H}^m(\pi) = \varepsilon^m \sum_\pi |\alpha_\pi| = 2c\mathbf{M}(T) \sum_\pi \underbrace{|\alpha_\pi|}_{\in \mathbb{N}} \leq c\mathbf{M}(T),$$

so $\alpha_\pi = 0$ for all π, and therefore $P = 0$. Finally, since $P = S = 0$, we have $\partial R = T$. □

To state the other application, we first give a definition.

Definition 4.100. The *flat distance* between m-currents $T_1, T_2 \in \mathcal{D}_m(\mathbb{R}^n)$ is

$$F(T_1,T_2) = \inf\{\mathbf{M}(S) + \mathbf{M}(R) \colon T_1 - T_2 = \partial R + S,\ R \in \mathcal{D}_{m+1}(\mathbb{R}^n), S \in \mathcal{D}_m(\mathbb{R}^n)\}.$$

Remark 4.101. $F(\cdot,\cdot)$ is a metric in $\{T \in \mathcal{D}_m(\mathbb{R}^n) \colon \mathbf{M}(T) < \infty\}$ and a convergence with respect to F is stronger than the weak convergence (i.e. convergence as currents):

$$F(T_i, T) \to 0 \Rightarrow T_i \to T,$$

but weaker than the mass convergence:

$$\mathbf{M}(T_i - T) \to 0 \Rightarrow F(T_i, T) \to 0.$$

Theorem 4.102 (Polyhedral approximation theorem)**.** *If $T \in \mathcal{D}_m(\mathbb{R}^n)$ with $\mathbf{M}(T) + \mathbf{M}(\partial T) < \infty$, there exists a sequence P_k of the form*

$$P_k = \sum_{\pi \in \mathcal{L}_{\varepsilon_k}, m} \alpha_\pi [\pi], \quad \alpha_\pi \in \mathbb{R},$$

such that $F(T, P_k) \to 0$ as $k \to \infty$. If $T \in \mathcal{R}_m(\mathbb{R}^n)$, we may choose $\alpha_\pi \in \mathbb{Z}$, so that $P_k \in \mathcal{P}_m(\mathbb{R}^n)$.

Proof. Applying the deformation theorem with $\varepsilon_k \searrow 0$, we get

$$T - P_k = \partial R_k + S_k,$$

where

$$\mathbf{M}(R_k) \le c\varepsilon_k \mathbf{M}(T) \to 0$$

and

$$\mathbf{M}(S_k) \le c\varepsilon_k \mathbf{M}(\partial T) \to 0,$$

and therefore

$$F(T, P_k) \le c\varepsilon_k \big(\mathbf{M}(T) + \mathbf{M}(\partial T)\big) \to 0$$

as $k \to \infty$. □

4.103 Rectifiability and compactness theorems

We say that a subset $D \subset X$ of a metric space X is *ε-dense,* $\varepsilon > 0$, if

$$X = \bigcup_{x \in \mathbb{D}} B(x, \varepsilon).$$

Furthermore, X is *totally bounded* if, for every $\varepsilon > 0$ there exists a *finite* ε-dense set $D \subset X$. Finally, recall that a metric space is compact if and only if it is complete and totally bounded.

We define the *flat norm* $F(T) = F(T, 0)$ for $T \in \mathcal{D}_m(\mathbb{R}^n)$, that is

$$F(T) = \inf\{\mathbf{M}(S) + \mathbf{M}(R) \colon T = \partial R + S,\ R \in \mathcal{D}_{m+1}(\mathbb{R}^n),\ S \in \mathcal{D}_m(\mathbb{R}^n)\}.$$

Thus

$$F(T_i) \to 0 \Rightarrow T_i \to 0.$$

The following converse holds for (integer multiplicity) rectifiable currents.

Theorem 4.104. *Suppose that* $T_0, T_j \in \mathcal{R}_m(\mathbb{R}^n)$*, with* $\operatorname{supp} T_j \subset K \subset \mathbb{R}^n$ *and* K *compact, and that*

$$\sup_j \{\mathbf{M}(T_j) + \mathbf{M}(\partial T_j)\} < \infty.$$

Then

$$T_j \to T_0 \iff F(T_j - T_0) \to 0.$$

Before the proof we first established the totally boundedness property: For every $\varepsilon > 0$ and $M > 0$ there exists $N = N(n, m, \varepsilon, M, K) \in \mathbb{N}$ such that

$$\{T \in \mathcal{R}_m(\mathbb{R}^n)\colon \operatorname{supp}(T) \subset K,\ \mathbf{M}(T) + \mathbf{M}(\partial T) < M\} \subset \bigcup_{j=1}^{N} B_F(R_j, \varepsilon) \tag{4.105}$$

for some $R_1, \ldots, R_N \in \mathcal{R}_m(\mathbb{R}^n)$, where

$$B_F(R, \varepsilon) = \{T \in \mathcal{R}_m(\mathbb{R}^n)\colon F(T - R) \leq \varepsilon\}.$$

Let $\delta > 0$ to be fixed later. By the deformation theorem there are $P, S \in \mathcal{R}_m(\mathbb{R}^n)$, $R \in \mathcal{R}_{m+1}(\mathbb{R}^n)$ such that

$$T - P = \partial R + S,$$

where

$$P = \sum_{\pi \in \mathcal{L}_{\delta,m}} \alpha_\pi [\pi], \quad \alpha_\pi \in \mathbb{Z}, \tag{4.106}$$

$$\mathbf{M}(P) = \sum_\pi |\alpha_\pi| \delta^m \leq c\mathbf{M}(T) \leq cM, \tag{4.107}$$

$$\operatorname{supp}(P) \subset \{x\colon \operatorname{dist}(x, K) < 2\delta\sqrt{n}\}, \tag{4.108}$$

$$\mathbf{M}(R) \leq c\delta M, \quad \mathbf{M}(S) \leq c\delta M. \tag{4.109}$$

Then

$$F(T - P) \leq \mathbf{M}(S) + \mathbf{M}(R) \leq 2c\delta M < \varepsilon$$

by choosing $\delta < \varepsilon/(2cM)$. On the other hand, there can be only finitely many, say at most N, currents P satisfying (4.106)-(4.108), where N depends only on K and $\delta = \delta(n, m, \varepsilon, M)$. This proves the local boundedness property (4.105).

Proof of Theorem 4.104. We need to prove the implication $T_j \to T \Rightarrow F(T_j - T) \to 0$. First we claim that the total boundedness property (4.105) implies that there is a subsequence T_{i_j} F-converging to $T_0' \in \mathcal{R}_m(\mathbb{R}^n)$, i.e. $F(T_{i_j} - T_0') \to 0$. Since $\{T_j\}$ belongs to a union of finitely many F-balls of radius 1, there exists $R \in \mathcal{R}_m(\mathbb{R}^n)$ such that $B_F(R, 1)$ contains infinitely many T_j's, call them $T_{1,j}$. Similarly, there exists another $R \in \mathcal{R}_m(\mathbb{R}^n)$ such that $B_F(R, 1)$ contains infinitely many $T_{1,j}$'s, call them $T_{2,j}$, and so on. Then the diagonal sequence $(T_{j,j})$ is a Cauchy sequence with respect to the F-norm. Passing to a subsequence, still denoted by $(T_{j,j})$, we may assume that

$$\sum_{j=2}^{\infty} F(T_{j,j} - T_{j-1,j-1}) < \infty,$$

where

$$T_{j,j} - T_{j-1,j-1} = \partial R_j + S_j, \quad R_j \in \mathcal{R}_{m+1}(\mathbb{R}^n),\ S_j \in \mathcal{R}_m(\mathbb{R}^n),$$

with

$$\sum_{j=2}^{\infty} \big[\mathbf{M}(R_j) + \mathbf{M}(S_j)\big] < \infty.$$

By Lemma 4.80

$$\sum_{j=2}^{\infty} R_j \in \mathcal{R}_{m+1}(\mathbb{R}^n), \quad \sum_{j=2}^{\infty} S_j \in \mathcal{R}_m(\mathbb{R}^n)$$

as limits of Cauchy sequences in the mass norm. Then for

$$T_0' := T_{1,1} + \sum_{j=2}^{\infty} S_j + \partial \sum_{j=2}^{\infty} R_j,$$

it holds that

$$F(T_0' - T_{j,j}) \le \sum_{i=j+1}^{\infty} \big[\mathbf{M}(R_j) + \mathbf{M}(S_j)\big] \to 0$$

as $j \to \infty$. Hence $T_j \to T_0'$, and so $T_0' = T_0$ and $F(T_{j,j} - T_0) \to 0$. Supposing that there exists a subsequence (T_{j_i}) such that $\liminf F(T_{j_i} - T_0) > 0$, we get a contradiction by repeating the argument above. Hence $F(T_j - T_0) \to 0$. □

Next we prove a rectifiability result whose proof uses the Besicovitch-Federer structure theorem (Theorem 2.55) on the characterization of purely m-unrectifiable set in terms of projections onto m-dimensional subspaces of $\mathbb{R}^n$. First we state the following consequence of Theorem 2.55, the proof is left as an exercise.

Lemma 4.110. *Let $E \subset \mathbb{R}^n$ be $\mathcal{H}^m$-measurable, with $\mathcal{H}^m(E) < \infty$. Suppose that E is purely m-unrectifiable. Then we can choose the coordinate axis such that $\mathcal{H}^m(P_I E) = 0$ for all $I \in \bigwedge(n, m)$.*

From Theorem 4.68 we then obtain the following:

Theorem 4.111. *Let E be as above and let $T \in \mathcal{D}_m(\mathbb{R}^n)$, with $\operatorname{supp} T$ compact and $\mathbf{M}(T) + \mathbf{M}(\partial T) < \infty$. Then $\mu_T(E) = 0$.*

Theorem 4.112 (Rectifiability theorem)**.** *Let $T \in \mathcal{D}_m(\mathbb{R}^n)$, with $\operatorname{supp} T$ compact and $\mathbf{M}(T) + \mathbf{M}(\partial T) < \infty$. If*

$$\Theta^{*m}(\mu_T, x) = \limsup_{r \searrow 0} \frac{\mu_T\big(\bar{B}(x, r)\big)}{\omega_m r^m} > 0$$

for μ_T-a.e. $x \in \mathbb{R}^n$, then there exist a countably m-rectifiable Borel set E and a Borel function $\theta\colon \mathbb{R}^n \to [0, +\infty]$ such that $\theta = 0$ on $\mathbb{R}^n \setminus E$,

$$T(\omega) = \int_E \langle \omega, \vec{T} \rangle \theta \, d\mathcal{H}^m$$

for $\omega \in \mathcal{D}^m(\mathbb{R}^n)$ and, for $\mathcal{H}^m$-a.e. $x \in E$, $\vec{T}(x)$ is a unit m-vector associated with the approximate (μ_T, m)-tangent space $V_x \in G(n, m)$ of E at x.

Proof. It follows from Theorem 4.29 that there exists a Radon measure μ_T on $\mathbb{R}^n$ and a μ_T-measurable mapping $\vec{T}\colon \mathbb{R}^n \to \bigwedge_m(\mathbb{R}^n)$ such that $|\vec{T}(x)| = 1$ for μ_T-a.e. $x \in \mathbb{R}^n$ and

$$T(\omega) = \int \langle \vec{T}(x), \omega(x)\rangle d\mu_T(x) \quad \forall \omega \in \mathcal{D}^m(\mathbb{R}^n).$$

Then main steps of the proof then are to establish that

(1) the set $\{x \in \mathbb{R}^n \colon \Theta^{*m}(\mu_T, x) > 0\}$ is countably m-rectifiable,

(2) $\mu_T \ll \mathcal{H}^m \llcorner E$, and that

(3) $\vec{T}\colon E \to \bigwedge_m(\mathbb{R}^n)$ is a Borel orientation, i.e. $\vec{T}(x) = \tau_1 \wedge \cdots \wedge \tau_m$ for $\mathcal{H}^m$-a.e. $x \in E$, where $\tau_1, \ldots, \tau_m$ is an orthonormal basis of the approximate (μ_T, m)-tangent space of E at x.

Using the Besicovitch covering theorem we can compare an arbitrary Radon measure μ and $\mathcal{H}^m$ (see e.g. [Si, p. 26], [Ma, 2.13], [Ho, 5.23]). Indeed, for all $A \subset \mathbb{R}^n$ and $\lambda > 0$

$$\mathcal{H}^m\big(\{x \in A \colon \Theta^{*m}(\mu, x) > \lambda\}\big) \le \lambda^{-1}\mu(A) \le \lambda^{-1}\mu(\mathbb{R}^n) \tag{4.113}$$

and

$$\mu\big(\{x \in A \colon \Theta^{*m}(\mu, x) < \lambda\}\big) \le \lambda \mathcal{H}^m(A). \tag{4.114}$$

From (4.113) and the assumption $\mathbf{M}(T) + \mathbf{M}(\partial T) < \infty$ we then obtain

$$\mathcal{H}^m\big(\{x \in \mathbb{R}^n \colon \Theta^{*m}(\mu_T, x) = \infty\}\big) = 0 = \mathcal{H}^m\big(\{x \in \mathbb{R}^n \colon \Theta^{*m}(\mu_{\partial T}, x) = \infty\}. \tag{4.115}$$

This together with Theorem 4.68 then implies that

$$\mu_T\big(\{x \in \mathbb{R}^n \colon \Theta^{*m}(\mu_T, x) = \infty\}\big) = 0 = \mu_{\partial T}\big(\{x \in \mathbb{R}^n \colon \Theta^{*m}(\mu_{\partial T}, x) = \infty\}. \tag{4.116}$$

Notice that since projections P_I are 1-Lipschitz, $\mathcal{H}^m(P_I A) = 0$ if $\mathcal{H}^m(A) = 0$. Define

$$E = \{x \in \mathbb{R}^n \colon \Theta^{*m}(\mu_T, x) > 0\}.$$

By (4.113), E has a σ-finite $\mathcal{H}^m$-measure. To prove that E is countably m-rectifiable, let $P \subset E$ be purely m-unrectifiable. By Lemma 4.110 and Theorem 4.111, we get $\mu_T(P) = 0$, and hence $\mathcal{H}^m(P) = 0$ by (4.113). So, E is countably m-rectifiable. By the definition of E, $\mu_T(\mathbb{R}^n \setminus E) = 0$, hence $T = T\llcorner E$, that is

$$T(\omega) = \int_E \langle \omega, \vec{T}\rangle d\mu_T, \ \forall \omega \in \mathcal{D}^m(\mathbb{R}^n).$$

By (4.114) and (4.116), we then conclude that $\mu_T \ll \mathcal{H}^m \llcorner E$, and therefore there exists, by the Radon-Nikodym theorem, a Borel function $\theta\colon E \to [0, +\infty]$ such that

$$\mu_T(A) = \int_A \theta d\mathcal{H}^m$$

for every Borel set $A \subset \mathbb{R}^n$. Hence

$$T(\omega) = \int_E \langle \omega, \vec{T}\rangle \theta d\mathcal{H}^m, \ \forall \omega \in \mathcal{D}^m(\mathbb{R}^n).$$

It remains to show that $\vec{T}$ is associated with the approximate (μ_T, m)-tangent space V_x of E at $\mathcal{H}^m$-a.e. $x \in E$. The approximate (μ_T, m)-tangent space at x is the m-dimensional subspace $V_x \in G(n,m)$ such that for every $\delta > 0$

$$\lim_{r \searrow 0} r^{-m} \mu_T\big(E \cap \bar{B}(x,r) \setminus \{y\colon\ \mathrm{dist}(y-x, V_x) < \delta|y-x|\} = 0.$$

We write E as a disjoint union

$$E = E_0 \sqcup \bigsqcup_{j=1}^{\infty} E_j,$$

where $\mathcal{H}^m(E_0) = 0$ and $E_j \subset M_j$, M_j being an m-dimensional C^1-smooth submanifold of $\mathbb{R}^n$. Then, in fact, $V_x = T_x^m E = T_x M_j$ for $\mathcal{H}^m$-a.e. $x \in E \cap M_j$. For $a \in \mathbb{R}^n$ and $\lambda > 0$ we define (as in Remark 2.51 (c)) $\eta_{a,\lambda}\colon \mathbb{R}^n \to \mathbb{R}^n$,

$$\eta_{a,\lambda}(y) = \frac{y-a}{\lambda}.$$

Then for $\mathcal{H}^m$-a.e. $x \in E$, we have

$$\lambda^{-m} \eta_{x,\lambda\sharp}(\mathcal{H}^m \llcorner E) \rightharpoonup \mathcal{H}^m \llcorner V_x$$

as $\lambda \searrow 0$. This follows from the area formula since

$$\begin{aligned}
\lambda^{-m} \eta_{x,\lambda\sharp}(\mathcal{H}^m \llcorner E)(A) &:= \lambda^{-m}(\mathcal{H}^m \llcorner E)\big(\eta_{x,\lambda}^{-1} A\big) = \lambda^{-m} \mathcal{H}^m\big(E \cap \eta_{x,\lambda}^{-1} A\big) \\
&= \int_{E \cap \eta_{x,\lambda}^{-1} A} J^E_{\eta_{x,\lambda}}(y) d\mathcal{H}^m(y) = \mathcal{H}^m\big(\eta_{x,\lambda} E \cap A\big) \\
&= (\mathcal{H}^m \llcorner \eta_{x,\lambda})(A).
\end{aligned}$$

More generally, for $\mathcal{H}^m$-a.e. $x \in E$ and for every Borel function $\psi\colon E \to [0, +\infty]$, with

$$\int_E \psi d\mathcal{H}^m < \infty,$$

we get

$$\lambda^{-m} \eta_{x,\lambda\sharp}(\mathcal{H}^m \llcorner \psi) \rightharpoonup \psi(x) \mathcal{H}^m \llcorner V_x,$$

that is,

$$\lambda^{-m} \int_E \varphi\big(\eta_{x,\lambda}(y)\big) \psi(y) d\mathcal{H}^m(y) \to \psi(x) \int_{V_x} \varphi d\mathcal{H}^m \tag{4.117}$$

for every $\varphi \in C_0(\mathbb{R}^n)$. We apply (4.117) with $\psi(y) = \langle dx^I, \vec{T}(y) \rangle \theta(y)$ to obtain

$$\begin{aligned}
\lambda^{-m} \int_E \omega_I\big(\eta_{x,\lambda}(y)\big) \big\langle dx^I, \vec{T}(y) \big\rangle \theta(y) d\mathcal{H}^m(y) &\to \theta(x) \big\langle dx^I, \vec{T}(x) \big\rangle \int_{V_x} \omega_I(y) d\mathcal{H}^m(y) \\
&= \theta(x) \int_{V_x} \big\langle \omega_I(y) dx^I, \vec{T}(x) \big\rangle d\mathcal{H}^m(y)
\end{aligned}$$

for $\mathcal{H}^m$-a.e. $x \in E$ and for all component functions ω_I of

$$\omega = \sum_{I \in \bigwedge(n,m)} \omega_I dx^I \in \mathcal{D}^m(\mathbb{R}^n).$$

Next we observe that λ^{-m} is the Jacobian determinant of the mapping $\eta_{x,\lambda}|M_j \colon M_j \to \eta_{x,\lambda}M_j$ between m-dimensional C^1-smooth submanifolds. Hence

$$\eta^*_{x,\lambda}\big(\omega|\eta_{x,\lambda}M_j\big)(y) = \lambda^{-m}(\omega|M_j)\big(\eta_{x,\lambda}(y)\big),$$

and therefore

$$\int_{E\cap M_j} \langle \eta^*_{x,\lambda}\omega(y), \vec{T}(y)\rangle d\mathcal{H}^m(y) \to \theta(x)\int_{V_x}\langle\omega(y),\vec{T}(x)\rangle d\mathcal{H}^m(y).$$

We define $S_x \in \mathcal{D}_m(\mathbb{R}^n)$ by

$$S_x(\omega) = \theta(x)\int_{V_x}\langle\omega(y),\vec{T}(x)\rangle d\mathcal{H}^m(y),\ \forall\omega\in\mathcal{D}^m(\mathbb{R}^n),$$

and claim that $\partial S_x = 0$. For that purpose let $\omega \in \mathcal{D}^{m-1}(\mathbb{R}^n)$ and $R > 0$ such that $\operatorname{supp}\omega \subset B^n(0,R)$. Then $\operatorname{supp}\eta^*_{x,\lambda}\omega \subset B^n(x,\lambda R)$, and therefore

$$\begin{aligned}\big|\partial\eta_{x,\lambda\sharp}T(\omega)\big| = \big|\eta_{x,\lambda\sharp}\partial T(\omega)\big| &= \left|\int\left\langle \omega\circ\eta_{x,\lambda}, \textstyle\bigwedge_m d\eta_{x,\lambda}\overrightarrow{\partial T}\right\rangle d\mu_{\partial T}\right| \\ &\le \lambda^{1-m}\|\omega\|_\infty\mu_{\partial T}\big(B(x,\lambda R)\big)\to 0\end{aligned}$$

as $\lambda\to 0$ if $\Theta^{*m}(\mu_{\partial T},x)<\infty$ which happens for $\mathcal{H}^m$-a.e. $x\in E$ by (4.115). We have proven that

$$\eta_{x,\lambda\sharp}T \to S_x \quad\text{and}\quad \partial\eta_{x,\lambda\sharp}T\to 0$$

as $\lambda\to 0$, and therefore $\partial S_x = 0$. Finally, to show that $\vec{T}(x)$ orients V_x, we may assume without loss of generality that $V_x = \mathbb{R}^m\times\{0\}$. For $j\in\{m+1,\dots,n\}$ and $I = (i_1,\dots,i_{m-1})\in\bigwedge(n,m-1)$, let

$$\omega(y) = y^j\varphi(y)dy^I = y^j\varphi(y)dy^{i_1}\wedge\cdots\wedge dy^{i_{m-1}},$$

where $\varphi\in C_0^\infty(\mathbb{R}^n)$ is arbitrary. Then

$$d\omega = d\big(y^j\varphi(y)\big)\wedge dy^I = \varphi(y)dy^j\wedge dy^I + y^j d\varphi\wedge dy^I$$

and $y^j\equiv 0$ in $V_x = \mathbb{R}^m\times\{0\}$, and hence

$$\begin{aligned}0 = \partial S_x(\omega) = S_x(d\omega) &= \theta(x)\int_{V_x}\varphi(y)\big\langle\vec{T}(x), dy^j\wedge dy^I\big\rangle d\mathcal{H}^m(y) \\ &= \theta(x)\int_{V_x}\varphi(y)\big\langle\vec{T}(x), e_j\wedge e_{1_i}\wedge\cdots\wedge e_{i_m-1}\big\rangle d\mathcal{H}^m(y).\end{aligned}$$

Since $\varphi\in C_0^\infty(\mathbb{R}^n)$ is arbitrary, we conclude that

$$\langle\vec{T}(x), e_j\wedge e_I\rangle = 0$$

for every $j\in\{m+1,\dots,n\}$ and $I=(i_1,\dots,i_{m-1})\in\bigwedge(n,m-1)$. As $|\vec{T}(x)|=1$, this proves that

$$\vec{T}(x) = \pm e_1\wedge\cdots\wedge e_m.$$

□

Remark 4.118. 1. We notice that the approximate (μ_T, m)-tangent space V_x coincides with the approximate tangent space $T_x^m E$ for $\mathcal{H}^m$-a.e. $x \in E$. Hence $T = \tau(E, \theta, \vec{T})$.

2. The compactness assumption on $\operatorname{supp} T$ is not necessary. It suffices only to assume that $\mathbf{M}_W(T) + \mathbf{M}_W(\partial T) < \infty$ for all $W \Subset \mathbb{R}^n$.

The next lemma is a step towards the compactness theorem.

Lemma 4.119. *Suppose $T_j \in \mathcal{R}_m(\mathbb{R}^n)$, $\partial T_j \in \mathcal{R}_{m-1}(\mathbb{R}^n)$, $\operatorname{supp} T_j \subset K$, with $K \subset \mathbb{R}^n$ compact, and that*

$$\sup_j \big(\mathbf{M}(T_j) + \mathbf{M}(\partial T_j)\big) < \infty.$$

If $T_j \to T$, then $T \in \mathcal{R}_m(\mathbb{R}^n)$.

Proof. We prove the lemma by induction on m. The case $m = 0$ is trivial. Suppose that the lemma holds for rectifiable $(m-1)$-currents with integer multiplicity. Let $T_j \in \mathcal{R}_m(\mathbb{R}^n)$, $T_j \to T$, be a sequence satisfying the assumptions.

First we prove by using Theorem 4.112 that T is a rectifiable m-current. For that purpose we will show that $\Theta(\mu_T, x) > 0$ for μ_T-a.e. $x \in \mathbb{R}^n$. For every $x \in \mathbb{R}^n$, let ρ_x be the 1-Lipschitz function $\rho_x(y) = |y - x|$. By Theorem 4.88 (3),

$$\langle T_j, \rho_x, t\rangle \in \mathcal{R}_{m-1}(\mathbb{R}^n) \tag{4.120}$$

for a.e. $t \in \mathbb{R}$. Clearly,

$$\begin{aligned} \langle T_j, \rho_x, t\rangle &= (\partial T_j)\llcorner\big(\mathbb{R}^n \setminus \bar{B}(x,t)\big) - \partial\big(T_j\llcorner(\mathbb{R}^n \setminus \bar{B}(x,t))\big) \\ &\to (\partial T)\llcorner\big(\mathbb{R}^n \setminus \bar{B}(x,t)\big) - \partial\big(T\llcorner(\mathbb{R}^n \setminus \bar{B}(x,t))\big) \\ &= \langle T, \rho_x, t\rangle. \end{aligned} \tag{4.121}$$

Theorem 4.88 (2) and Fatou's lemma imply that, for all $\delta > 0$,

$$\int_\delta^\infty \liminf_{j\to\infty} \mathbf{M}(\langle T_j, \rho_x, t\rangle)dt \le \liminf_{j\to\infty} \int_\delta^\infty \mathbf{M}(\langle T_j, \rho_x, t\rangle)dt \le \liminf_{j\to\infty} \mathbf{M}(T_j),$$

and similarly,

$$\begin{aligned} \int_\delta^\infty \liminf_{j\to\infty} \mathbf{M}(\partial\langle T_j, \rho_x, t\rangle)dt &= \int_\delta^\infty \liminf_{j\to\infty} \mathbf{M}(\langle \partial T_j, \rho_x, t\rangle)dt \\ &\le \liminf_{j\to\infty} \int_\delta^\infty \mathbf{M}(\langle \partial T_j, \rho_x, t\rangle)dt \\ &\le \liminf_{j\to\infty} \mathbf{M}(\partial T_j). \end{aligned}$$

Hence for a.e. $t \in \mathbb{R}$ there exists a subsequence such that

$$\sup_{j_i} \big(\mathbf{M}(\langle T_{j_i}, \rho_x, t\rangle) + \mathbf{M}(\partial\langle T_{j_i}, \rho_x, t\rangle)\big) < \infty. \tag{4.122}$$

The induction hypothesis together with (4.120), (4.121), and (4.122) imply that

$$\langle T, \rho_x, t\rangle \in \mathcal{R}_{m-1}(\mathbb{R}^n) \tag{4.123}$$

for a.e. t. On the other hand, since $\partial T_j \to \partial T$, we get from the induction hypothesis that

$$(\partial T) \llcorner B(x,t) \in \mathcal{R}_{m-1}(\mathbb{R}^n), \tag{4.124}$$

and since

$$\langle T, \rho_x, t\rangle = \partial\big(T \llcorner B(x,t)\big) - (\partial T)\llcorner B(x,t) \tag{4.125}$$

for a.e. t by Theorem 4.87 (1), we obtain

$$\partial\big(T\llcorner B(x,t)\big) = \langle T, \rho_x, t\rangle + (\partial T)\llcorner B(x,t) \in \mathcal{R}_{m-1}(\mathbb{R}^n) \tag{4.126}$$

for a.e. t.

Next we want to reduce the proof to the case $\partial T = 0$. Combining Example 4.74 (2) and (3) we conclude that the cone over a rectifiable current with integer multiplicity is a rectifiable current with integer multiplicity, that is

$$S \in \mathcal{R}_d(\mathbb{R}^n) \Rightarrow 0 \lhd S = h_\sharp\big(\big[[0,1]\big] \times S\big) \in \mathcal{R}_{d+1}(\mathbb{R}^n), \quad h(t,x) = tx.$$

Hence $0 \lhd \partial T \in \mathcal{R}_m(\mathbb{R}^n)$ since $\partial T \in \mathcal{R}_{m-1}(\mathbb{R}^n)$ by the induction hypothesis. We also notice that (by the homotopy formula)

$$\partial\big(0 \lhd (\partial T) - T\big) = \partial(0 \lhd \partial T) - \partial T = \partial T - 0 \lhd \partial(\partial T) - \partial T = 0.$$

Hence we may assume without loss of generality that $\partial T = 0$. Indeed, otherwise we may consider the sequence $\tilde{T}_j = T_j - 0 \lhd \partial T_j \in \mathcal{R}_m(\mathbb{R}^n)$, with properties

$$\begin{aligned} \tilde{T}_j &= T_j - 0 \lhd \partial T_j \to \tilde{T} := T - 0 \lhd \partial T, \\ \partial \tilde{T}_j &= 0, \\ \mathbf{M}(\tilde{T}_j) &\le \mathbf{M}(T_j) + \mathbf{M}(\partial T_j). \end{aligned}$$

Define, for a fixed $x \in \mathbb{R}^n$,

$$f(r) = \mu_T\big(\bar{B}(x,r)\big).$$

Using the assumption $\partial T = 0$ we obtain from Theorem 4.87 (3) that

$$\mathbf{M}\big(\partial\big(T\llcorner B(x,r)\big)\big) = \mathbf{M}\big(\langle T, \rho_x, r\rangle\big) \le \liminf_{h \searrow 0} h^{-1}\big(f(r+h) - f(r)\big) = f'(r) \tag{4.127}$$

for a.e. $r > 0$. Suppose then that $\Theta^{*m}(\mu_T, x) < \eta < 1$, so that

$$\limsup_{s\to 0} \frac{f(s)}{\omega_m s^m} < \eta$$

and that

$$\frac{1}{\delta}\int_0^\delta \frac{d}{dr}\left(f^{1/m}(r)\right) dr \le \frac{1}{\delta} f^{1/m}(\delta) \le \omega_m^{1/m}\eta^{1/m}$$

for sufficiently small $\delta > 0$. Hence we have

$$\frac{d}{dr}\left(f^{1/m}(r)\right) = \tfrac{1}{m} f^{\frac{1}{m}-1}(r) f'(r) \le 2\omega_m^{1/m}\eta^{1/m},$$

or equivalently

$$f'(r) \le 2m\omega_m^{1/m}\eta^{1/m} f^{\frac{m-1}{m}}(r) \tag{4.128}$$

for all r in a subset of $[0,\delta]$ of positive m_1-measure. Suppose from now on that $m > 1$ (see Remark 4.134 for the case $n = 1$). By the isoperimetric inequality (Theorem 4.99) applied to $\partial\big(T\llcorner B(x,r)\big) \in \mathcal{R}_{m-1}(\mathbb{R}^n)$, there exists $S_r \in \mathbb{R}_m(\mathbb{R}^n)$ with the properties

$$\partial S_r = \partial\big(T\llcorner B(x,r)\big)$$

and

$$\mathbf{M}(S_r)^{\frac{m-1}{m}} \le C\mathbf{M}\big(\partial(T\llcorner B(x,r)\big) \le c\eta^{1/m}\mathbf{M}\big(T\llcorner \bar{B}(x,r)\big)^{\frac{m-1}{m}},$$

where also (4.127) and(4.128) were used. Thus there exists a sequence $r_i \searrow 0$ such that

$$\partial S_{r_i} = \partial\big(T\llcorner \bar{B}(x,r_i)\big) \in \mathcal{R}_{m-1}(\mathbb{R}^n)$$

and

$$\mathbf{M}(S_{r_i}) \le c\eta^{\frac{1}{m-1}}\mathbf{M}\big(T\llcorner \bar{B}(x,r_i)\big).$$

Let then $C \subset \{x\colon \Theta^{*m}(\mu_T,x) < \eta\}$ be compact. By Vitali's covering theorem for the Radon measure μ_T, we find, for all $\varrho > 0$, disjoint balls $B_j^\varrho = \bar{B}(x_j,r_j)$ such that $x_j \in C,\ r_j < \varrho$,

$$\mu_T\big(C \setminus \bigcup_j B_j^\varrho\big) = 0, \tag{4.129}$$

$$B_j^\varrho \subset \{x\colon \operatorname{dist}(x,C) < \varrho\}, \tag{4.130}$$

$$\mathbf{M}(S_j^\varrho) \le c\eta^{\frac{1}{m-1}}\mathbf{M}\big(T\llcorner B_j^\varrho\big) \tag{4.131}$$

for some $S_j^\varrho \in \mathcal{R}_m(\mathbb{R}^n)$ with

$$\partial S_j^\varrho = \partial\big(T\llcorner B_j^\varrho\big).$$

By the homotopy formula, with $h(t,x) = tx + (1-t)x_j$, we then have

$$S_j^\varrho - T\llcorner B_j^\varrho = \partial\big(x_j \lhd (S_j^\varrho - T\llcorner B_j^\varrho)\big)$$

and hence by (4.46) and (4.131)

$$\begin{aligned}
\big|\big(S_j^\varrho - T\llcorner B_j^\varrho\big)(\omega)\big| &= \big|\big(x_j \lhd (S_j^\varrho - T\llcorner B_j^\varrho)\big)(d\omega)\big| \\
&\overset{(4.46)}{\le} \varrho\mathbf{M}\big(S_j^\varrho - T\llcorner B_j^\varrho\big)\|d\omega\|_\infty \\
&\le c\varrho(\eta^{\frac{1}{m-1}}+1)\mathbf{M}(T\llcorner B_j^\varrho)\|d\omega\|_\infty.
\end{aligned}$$

Since the balls B_j^ϱ are disjoint and $\mathbf{M}(T) < \infty$, we get

$$\sum_j \big(S_j^\varrho - T\llcorner B_j^\varrho\big) \to 0 \quad \text{as } \varrho \to 0, \tag{4.132}$$

and so

$$T\llcorner C = \lim_{\varrho\to 0}\sum_j T\llcorner B_j^\varrho = \lim_{\varrho\to 0}\sum_j S_j^\varrho$$

by (4.129), (4.130), and (4.132). It then follows that

$$\begin{aligned}
\mu_T(C) = \mathbf{M}(T\llcorner C) &\le \liminf_{\varrho\to 0}\mathbf{M}\big(\sum_j S_j^\varrho\big) \le \liminf_{\varrho\to 0}\sum_j \mathbf{M}(S_j^\varrho) \\
&\le \liminf_{\varrho\to 0} c\eta^{\frac{1}{m-1}}\sum_j \mathbf{M}\big(T\llcorner B_j^\varrho\big) \le \liminf_{\varrho\to 0} c\eta^{\frac{1}{m-1}}\sum_j \mu_T(B_j^\varrho) \\
&= \liminf_{\varrho\to 0} c\eta^{\frac{1}{m-1}}\mu_T\big(\bigcup_j B_j^\varrho\big) \\
&= c\eta^{\frac{1}{m-1}}\mu_T(C).
\end{aligned}$$

If $c\eta^{\frac{1}{m-1}} < 1$, we obtain $\mu_T(C) = 0$. Hence

$$\Theta^{*m}(\mu_T, x) > 0$$

for μ_T-a.e. $x \in \mathbb{R}^n$. By Theorem 4.112, $T = \tau(E, \theta, \vec{T})$, where $E \subset \mathbb{R}^n$ is a countably m-rectifiable Borel set and $\theta\colon E \to [0, \infty]$ is a Borel function such that

$$T(\omega) = \int_E \langle \omega, \vec{T} \rangle \theta \, d\mathcal{H}^m, \quad \forall \omega \in \mathcal{D}^m(\mathbb{R}^n).$$

It remains to show that θ is integer valued. Then $\mathcal{H}^m(E) < \infty$ since

$$\mathbf{M}(T) = \int_E \theta \, d\mathcal{H}^m < \infty.$$

As in the proof of Theorem 4.112, we have

$$\eta_{x,r\sharp} T \to \theta(x)[V_x] \quad \text{as } r \to 0$$

for $\mathcal{H}^m$-a.e. $x \in E$, where $\eta_{x,r}(y) = (y - x)/r$ and $V_x = T_x^m E$ is the approximate (μ_T, m)-tangent space of E at x. Fixing such x, we may assume that

$$V_x = \mathbb{R}^m \times \{0\} = \mathbb{R}^m.$$

Let $P\colon \mathbb{R}^n \to \mathbb{R}^m$ be the projection $P(x, y) = x$. Since

$$P_\sharp\big(\partial(\eta_{x,r\sharp} T_j)\big) = P_\sharp\big(\eta_{x,r\sharp}(\partial T_j)\big) \in \mathcal{R}_{m-1}(\mathbb{R}^n)$$

by the assumption $\partial T_j \in \mathcal{R}_{m-1}(\mathbb{R}^n)$ and Example 4.74 (3), and since

$$P_\sharp\big(\partial(\eta_{x,r\sharp} T_j)\big) \to P_\sharp\big(\partial(\eta_{x,r\sharp} T)\big),$$

we have

$$\partial P_\sharp\big(\eta_{x,r\sharp} T\big) = P_\sharp\big(\partial(\eta_{x,r\sharp} T)\big) \in \mathcal{R}_{m-1}(\mathbb{R}^n)$$

by the induction hypothesis. We conclude (see Lemma 4.133 below) that

$$P_\sharp\big(\eta_{x,r\sharp} T\big) \in \mathcal{R}_m(\mathbb{R}^m).$$

By Theorem 4.65 there exist integer valued functions $g_r \in BV(\mathbb{R}^m)$ such that

$$P_\sharp\big(\eta_{x,r\sharp} T\big)(\omega) = \int_{\mathbb{R}^m} \varphi g_r \, d\mathcal{H}^m = \int_{\mathbb{R}^m} \langle \omega, e_1 \wedge \cdots \wedge e_m \rangle g_r \, d\mathcal{H}^m$$

for $\omega = \varphi dx^1 \wedge \cdots \wedge dx^m \in \mathcal{D}^m(\mathbb{R}^m)$. But

$$P^*(\omega) = (\varphi \circ P) dx^1 \wedge \cdots \wedge dx^m \in \mathcal{D}^m(\mathbb{R}^n),$$

and so

$$\int_{\mathbb{R}^m} \varphi g_r \, d\mathcal{H}^m = \eta_{x,r\sharp} T(P^*\omega) \to \theta(x)[\mathbb{R}^m](P^*\omega) = \theta(x) \int_{\mathbb{R}^m} \varphi d\mathcal{H}^m$$

as $r \to 0$. Since all g_r's are integer valued, we conclude that $\theta(x) \in \mathbb{Z}$ which proves the lemma. □

Lemma 4.133. *If $S \in \mathcal{D}_m(\mathbb{R}^m)$, $\operatorname{supp} S$ is compact, and $\partial S \in \mathcal{R}_{m-1}(\mathbb{R}^m)$, then $S \in \mathcal{R}_m(\mathbb{R}^m)$.*

Proof. Since $0 \lhd S \in \mathcal{D}_{m+1}(\mathbb{R}^m) = \{0\}$, we have

$$s = 0 \lhd \partial S + \partial(0 \lhd S) = 0 \lhd \partial S = h_\sharp\big([[0,1]] \times \partial S\big) \in \mathcal{R}_m(\mathbb{R}^m)$$

by Example 4.74 (2), (3). □

Remark 4.134. In the case $m = 1$ (in Lemma 4.119), we have

$$\partial\big(T \llcorner B(x,t)\big) \in \mathcal{R}_0(\mathbb{R}^n)$$

for a.e. t by (4.126). Assuming $\partial\big(T \llcorner B(x,t)\big) \neq 0$, we get a contradiction

$$1 \leq \mathbf{M}\Big(\partial\big(T \llcorner B(x,r)\big)\Big) \leq f'(r) \leq 4\eta < 1$$

by (4.127) and (4.128), with $\eta < 1/4$. Hence $\partial\big(T \llcorner B(x,t)\big) = 0$ for a.e. r and we may take $S_r = 0$ (and thus $S_j^\varrho = 0$ in (4.131)). It follows that $\mu_T(C) = 0$.

Theorem 4.135 (Boundary rectifiability theorem)**.** *Let $T \in \mathcal{R}_m(\mathbb{R}^n)$ with* $\operatorname{supp} T$ *compact and* $\mathbf{M}(\partial T) < \infty$*. Then $\partial T \in \mathcal{R}_{m-1}(\mathbb{R}^n)$.*

Proof. By the polyhedral approximation theorem 4.102, there exists a sequence $P_k \in \mathcal{P}_m(\mathbb{R}^n)$ of the form

$$P_k = \sum_{\pi \in \mathcal{L}_{\varepsilon_k}, m} \alpha_\pi [\pi], \quad \alpha_\pi \in \mathbb{Z},$$

such that $P_k \to T$ and $\partial P_k \to \partial T$ as $k \to \infty$. Since $\partial P_k \in \mathcal{R}_{m-1}(\mathbb{R}^n)$, and $\partial(\partial P_k) = 0$, we conclude from Lemma 4.119 that

$$\partial T = \lim_k \partial P_k \in \mathcal{R}_{m-1}(\mathbb{R}^n).$$

□

Theorem 4.136 (Compactness theorem)**.** *Suppose that $T_j \in \mathcal{R}_m(\mathbb{R}^n)$, with* $\operatorname{supp}(T_j) \subset K$ *and $K \subset \mathbb{R}^n$ compact, and that*

$$\sup_j \big\{\mathbf{M}(T_j) + \mathbf{M}(\partial T_j)\big\} < \infty.$$

Then there exist a subsequence T_{j_i} and $T \in \mathcal{R}_m(\mathbb{R}^n)$ such that $T_{j_i} \to T$.

Proof. By Theorem 4.135, $\partial T_j \in \mathcal{R}_{m-1}(\mathbb{R}^n)$ and the claim then follows from Lemma 4.119. □

5 Mass minimizing currrents

In this final section we discuss briefly mass (area) minimizing currents that provide a tool to attack the general Plateau problem. In particular, we prove the existence of a mass minimizing integer multiplicity rectifiable m-current given a rectifiable $(m-1)$-cycle (of integer multiplicity).

Definition 5.1. An m-current $S \in \mathcal{R}_m(\mathbb{R}^n)$ is *mass minimizing* if

$$\mathbf{M}(S) \leq \mathbf{M}(T)$$

for every $T \in \mathcal{R}_m(\mathbb{R}^n)$ with $\partial T = \partial S$.

Theorem 5.2 (Existence theorem). *If $T \in \mathcal{R}_{m-1}(\mathbb{R}^n)$ with $\partial T = 0$ and* $\operatorname{supp}(T)$ *is compact, there exists $S_0 \in \mathcal{R}_m(\mathbb{R}^n)$ such that $\partial S_0 = T$ and*

$$\mathbf{M}(S_0) = \min\{\mathbf{M}(S) \colon S \in \mathcal{R}_m(\mathbb{R}^n),\ \partial S = T\}.$$

Hence S_0 is mass minimizing.

Proof. By the isoperimetric inequality (Theorem 4.99) there is a m-current $S \in \mathcal{R}_m(\mathbb{R}^n)$ with $\operatorname{supp}(S)$ compact such that $\partial S = T$ and

$$\mathbf{M}(S)^{m/(m+1)} \le C_{n,m}\mathbf{M}(T) < \infty.$$

Hence the set $\mathcal{S} = \{S \in \mathcal{R}_m(\mathbb{R}^n) \colon \partial S = T\}$ is non-empty (in fact, also $0 \lhd T$ would do) and we may find a minimizing sequence $S_j \in \mathcal{R}_m(\mathbb{R}^n),\ \partial S_j = T$, such that

$$\mathbf{M}(S_j) \to I := \inf\{\mathbf{M}(S) \colon S \in \mathcal{R}_m(\mathbb{R}^n),\ \partial S = T\}.$$

Let $R > 0$ be so large that $\operatorname{supp}(T) \subset \bar{B}(0,R)$ and let $f \colon \mathbb{R}^n \to \bar{B}(0,R)$,

$$f(x) = \begin{cases} Rx/|x|, & \text{if } |x| \ge R; \\ x, & \text{if } |x| \le R, \end{cases}$$

be the 1-Lipschitz retract onto $\bar{B}(0,R)$. Then $\mathbf{M}(f_\sharp S) \le \mathbf{M}(S)$ for every $S \in \mathcal{D}_m(\mathbb{R}^n)$ by (4.45). Hence we may assume that $\operatorname{supp}(S_j) \subset \bar{B}(0,R)$ for every j. Moreover

$$\sup_j\big\{\mathbf{M}(S_j) + \mathbf{M}(\partial S_j)\big\} = \sup_j\big\{\mathbf{M}(S_j) + \mathbf{M}(T)\big\} < \infty.$$

By the compactness theorem (Theorem 4.136) there exists a subsequence S_{j_i} and $S_0 \in \mathcal{R}_m(\mathbb{R}^n)$ such that $S_{j_i} \to S_0$. Then $T = \partial S_{j_i} \to \partial S_0$, and therefore $\partial S_0 = T$ and

$$I \le \mathbf{M}(S_0) \le \liminf_{j_i \to \infty} \mathbf{M}(S_{j_i}) = I.$$

□

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6 Appendix

6.1 Proof of Riesz' representation theorem 1.62

First we prove the auxiliary lemma (Lemma 1.63).

Proof of Lemma 1.63. (a) Write $\delta = \operatorname{dist}(K, \partial V)$. Because K is compact, it follows that $\delta > 0$. Then the function

$$f(x) = \max\big(0, 1 - \frac{2}{\delta}\operatorname{dist}(x, K)\big)$$

satisfies the conditions of part (a).

(b) For every $x \in K$ there exists a ball $B(x, r_x)$, with $B(x, 2r_x) \subset V_j$ for some j. Because K is compact, it can be covered by finitely many such balls, i.e.

$$K \subset \bigcup_{i=1}^{k} B(x_i, r_{x_i}).$$

Let A_j be the union of those closed balls $\bar{B}(x_i, r_{x_i})$ for which $B(x_i, 2r_{x_i}) \subset V_j$. Then

$$A_j \subset V_j \quad \text{and} \quad K \subset \bigcup_{j=1}^{m} A_j.$$

By part (a) we choose functions $g_j \in C_0(\mathbb{R}^n)$ s.t.

$$\chi_{A_j} \leq g_j \leq 1 \quad \text{and} \quad \operatorname{supp}(g_j) \subset V_j.$$

Then define

$$\begin{aligned} h_1 &= g_1, \\ h_2 &= (1 - g_1)g_2, \\ &\vdots \\ h_m &= (1 - g_1) \cdots (1 - g_{m-1})g_m. \end{aligned}$$

Then clearly

$$0 \leq h_j \leq 1 \quad \text{and} \quad \operatorname{supp}(h_j) \subset V_j.$$

Induction with respect to m shows that $\sum_{j=1}^{m} h_j = 1 - (1 - g_1) \cdots (1 - g_m)$. Furthermore,

$$\chi_K \leq \sum_{j=1}^{m} h_j \leq 1,$$

because if $x \in K$, then $x \in A_j$ for some j and consequently $1 - g_j(x) = 0$ and $\sum_{j=1}^{m} h_j(x) = 1$. □

Remark 6.2. In the case of a locally compact Hausdorff space Lemma 1.63 is (a version of) Urysohn's lemma.

Proof of Riesz' representation theorem 1.62. Define

$$\tilde{\mu}(\emptyset) = 0$$

and set

$$\tilde{\mu}(V) = \sup\{\Lambda(f)\colon f \in C_0(\mathbb{R}^n),\ 0 \le f \le 1 \text{ and } \operatorname{supp}(f) \subset V\}$$

for every *open* set $V \subset \mathbb{R}^n$. Then it follows from the definition that

$$0 \le \tilde{\mu}(V_1) \le \tilde{\mu}(V_2), \tag{6.3}$$

if $V_1, V_2 \subset \mathbb{R}^n$ are open and $V_1 \subset V_2$.

Next define

$$\tilde{\mu}(A) = \inf\{\tilde{\mu}(V)\colon A \subset V \subset \mathbb{R}^n,\ V \text{ open}\} \tag{6.4}$$

for all $A \subset \mathbb{R}^n$. We show that $\tilde{\mu}$ is a *metric* outer measure. Then all Borel-sets of $\mathbb{R}^n$ will be $\tilde{\mu}$-measurable by Theorem 1.18.

1. *Monotonicity*

   $$\tilde{\mu}(A_1) \le \tilde{\mu}(A_2), \quad \text{if } A_1 \subset A_2,$$

   follows directly from (6.3) and the definition (6.4).

2. We prove first *subadditivity* for open sets. In other words, if $V_j \subset \mathbb{R}^n,\ j \in \mathbb{N}$, are open, then

   $$\tilde{\mu}\Big(\bigcup_{j=1}^{\infty} V_j\Big) \le \sum_{j=1}^{\infty} \tilde{\mu}(V_j). \tag{6.5}$$

   To prove this let $f \in C_0(\mathbb{R}^n)$ s.t. $0 \le f \le 1$ and $\operatorname{supp}(f) \subset \bigcup_{j=1}^{\infty} V_j$. Because of the compactness of $\operatorname{supp}(f)$

   $$\operatorname{supp}(f) \subset \bigcup_{j=1}^{m} V_j.$$

   Write $K = \operatorname{supp}(f)$. Lemma 1.63, part (b), implies that there exist functions $h_j \in C_0(\mathbb{R}^n)$ with

   $$0 \le h_j \le 1, \quad \operatorname{supp}(h_j) \subset V_j \quad \text{and} \quad \chi_K \le \sum_{j=1}^{m} h_j \le 1.$$

   Then

   $$f = \sum_{j=1}^{m} h_j f,$$

   $$\operatorname{supp}(h_j f) \subset V_j \quad \text{and} \quad 0 \le h_j f \le 1 \quad \forall\, j = 1, \dots, m,$$

   and hence

   $$\Lambda(f) = \sum_{j=1}^{m} \Lambda(h_j f) \le \sum_{j=1}^{m} \tilde{\mu}(V_j) \le \sum_{j=1}^{\infty} \tilde{\mu}(V_j).$$

   Taking sup over all "admissible" functions f in the definition of $\tilde{\mu}\big(\cup_j V_j\big)$ we obtain (6.5). Let then $A_j \subset \mathbb{R}^n, j \in \mathbb{N}$, be arbitrary sets. Fix $\varepsilon > 0$ and choose open sets $V_j \subset \mathbb{R}^n$ s.t. $A_j \subset V_j$ and

   $$\tilde{\mu}(V_j) \le \tilde{\mu}(A_j) + \varepsilon/2^j.$$

Then

$$\bigcup_{j=1}^{\infty} A_j \subset \bigcup_{j=1}^{\infty} V_j,$$

and hence by monotonicity and (6.5) we obtain

$$\tilde{\mu}\big(\bigcup_{j=1}^{\infty} A_j\big) \le \tilde{\mu}\big(\bigcup_{j=1}^{\infty} V_j\big) \le \sum_{j=1}^{\infty} \tilde{\mu}(V_j) \le \sum_{j=1}^{\infty} \tilde{\mu}(A_j) + \varepsilon,$$

which implies subadditivity for all sets by letting $\varepsilon \to 0$. We have proved that $\tilde{\mu}$ is an outer measure.

3. Let $V_1, V_2 \subset \mathbb{R}^n$ be open sets and $\operatorname{dist}(V_1, V_2) > 0$. Let further $f_j \in C_0(\mathbb{R}^n)$ s.t. $0 \le f_j \le 1$ and $\operatorname{supp}(f_j) \subset V_j,\ j = 1, 2$. Then $0 \le f_1 + f_2 \le 1$ and $\operatorname{supp}(f_1 + f_2) \subset V_1 \cup V_2$, and hence

$$\Lambda(f_1) + \Lambda(f_2) = \Lambda(f_1 + f_2) \le \tilde{\mu}(V_1 \cup V_2).$$

Taking sup over all admissible functions f_1 and f_2 we obtain

$$\tilde{\mu}(V_1) + \tilde{\mu}(V_2) \le \tilde{\mu}(V_1 \cup V_2) \overset{(6.5)}{\le} \tilde{\mu}(V_1) + \tilde{\mu}(V_2). \tag{6.6}$$

Let then $A_1, A_2 \subset \mathbb{R}^n$ be arbitrary sets with $\operatorname{dist}(A_1, A_2) > 0$. Fix $\varepsilon > 0$ and choose an open set $V \subset \mathbb{R}^n$ s.t. $A_1 \cup A_2 \subset V$ and

$$\tilde{\mu}(V) \le \tilde{\mu}(A_1 \cup A_2) + \varepsilon.$$

Choose then open sets $V_j \subset \mathbb{R}^n,\ j = 1, 2$, s.t. $A_j \subset V_j$ and $\operatorname{dist}(V_1, V_2) > 0$. (We may choose for instance $V_j = \{x \in \mathbb{R}^n \colon \operatorname{dist}(x, A_j) < \frac{1}{3} \operatorname{dist}(A_1, A_2)\}$.) Now $A_j \subset V_j \cap V$ and

$$\operatorname{dist}(V_1 \cap V, V_2 \cap V) > 0,$$

and hence by (6.6)

$$\begin{aligned}
\tilde{\mu}(A_1) + \tilde{\mu}(A_2) &\le \tilde{\mu}(V_1 \cap V) + \tilde{\mu}(V_2 \cap V) \\
&= \tilde{\mu}(V \cap (V_1 \cup V_2)) \\
&\le \tilde{\mu}(V) \\
&\le \tilde{\mu}(A_1 \cup A_2) + \varepsilon.
\end{aligned}$$

Letting now $\varepsilon \to 0$ we see that $\tilde{\mu}$ is a metric outer measure.

4. We prove next that $\tilde{\mu}$ is locally finite: If $B(x, r) \subset \mathbb{R}^n$, then choose $f_0 \in C_0(\mathbb{R}^n)$ s.t. $0 \le f_0 \le 1$ and

$$\chi_{B(x,r)} \le f_0.$$

Then $f \le f_0$ for all functions f admissible in the definition of $\tilde{\mu}\big(B(x, r)\big)$. Because $\Lambda(f_0 - f) \ge 0$, then

$$\Lambda(f) \le \Lambda(f_0).$$

Taking sup over all such functions f we obtain

$$\tilde{\mu}\big(B(x, r)\big) = \sup_f \Lambda(f) \le \Lambda(f_0) < \infty.$$

Corollary 1.32 implies that

$$\mu = \tilde{\mu} | \operatorname{Bor}(\mathbb{R}^n)$$

is a Radon measure.

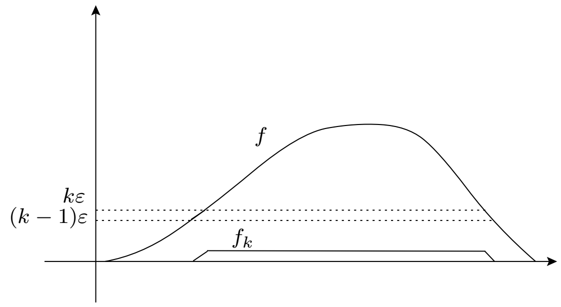
5. We must still show that

$$\Lambda(f) = \int_{\mathbb{R}^n} f\, d\mu$$

for all $f \in C_0(\mathbb{R}^n)$.

Let $f \in C_0(\mathbb{R}^n)$. We may suppose that $f \geq 0$, because $f = f_+ - f_-$, where $f_+ = \max(0, f) \in C_0(\mathbb{R}^n)$ and $f_- = \max(0, -f) \in C_0(\mathbb{R}^n)$. Fix $\varepsilon > 0$ and set for all $k \in \mathbb{N}$

$$f_k(x) = \max\big((k-1)\varepsilon, \min(f(x), k\varepsilon)\big) - (k-1)\varepsilon.$$



Clearly $0 \leq f_k \leq \varepsilon$ and $f_k \in C_0(\mathbb{R}^n)$ for all k. Because $f_k \equiv 0$, if $(k-1)\varepsilon \geq \|f\|_\infty$, then

$$f = \sum_{k=1}^{m} f_k$$

for some $m \in \mathbb{N}$. Let $K(k) = \{x \in \mathbb{R}^n \colon f(x) \geq k\varepsilon\},\ k \in \mathbb{N}$ and $K(0) = \operatorname{supp}(f)$. Then

$$\varepsilon \chi_{K(k)} \leq f_k \leq \varepsilon \chi_{K(k-1)}, \tag{6.7}$$

and hence

$$\varepsilon \sum_{k=1}^{m} \mu\big(K(k)\big) \leq \sum_{k=1}^{m} \int_{\mathbb{R}^n} f_k\, d\mu = \int_{\mathbb{R}^n} f\, d\mu \leq \varepsilon \sum_{k=1}^{m} \mu\big(K(k-1)\big). \tag{6.8}$$

On the other hand, if $\delta > 0$, then by (6.7)

$$\frac{1}{\varepsilon}(1+\delta) f_k \geq 1$$

in some neighbourhood W of $K(k)$. In particular,

$$\frac{1}{\varepsilon}(1+\delta) f_k \geq g$$

for every function $g \in C_0(\mathbb{R}^n)$ admissible in the definition of $\mu(W)$. Thus

$$\Lambda\big((1+\delta) f_k\big) \geq \varepsilon\mu(W) \geq \varepsilon\mu\big(K(k)\big),$$

and further

$$\Lambda(f_k) \geq \varepsilon\mu\big(K(k)\big)$$

letting $\delta \to 0$. In the same way, f_k/ε is admissible in the definition of $\mu(V)$ for every neighbourhood V of $K(k-1)$, and hence $\Lambda(f_k) \leq \varepsilon\mu(V)$. Then by the definition

$$\Lambda(f_k) \leq \varepsilon\mu\big(K(k-1)\big).$$

Combining these inequalities we obtain

$$\sum_{k=1}^{m} \varepsilon\mu\big(K(k)\big) \le \Lambda(f) \le \varepsilon \sum_{k=1}^{m} \mu\big(K(k-1)\big). \tag{6.9}$$

The inequalities (6.8) and (6.9) imply that

$$\begin{aligned} \Big|\Lambda(f) - \int_{\mathbb{R}^n} f\,d\mu\Big| &\le \varepsilon \sum_{k=1}^{m} \big(\mu\big(K(k-1)\big) - \mu\big(K(k)\big)\big) \\ &= \varepsilon \left(\mu\big(K(0)\big) - \mu\big(K(m)\big)\right) \\ &\le \varepsilon \underbrace{\mu\big(\operatorname{supp}(f)\big)}_{<\infty}. \end{aligned}$$

Letting $\varepsilon \to 0$ we see that

$$\Lambda(f) = \int_{\mathbb{R}^n} f\,d\mu$$

and thus μ is the desired Radon measure.

6. Finally, we prove the uniqueness of μ. Let μ_1 also be a Radon-measure, for which

$$\Lambda(f) = \int_{\mathbb{R}^n} f\,d\mu_1$$

for all $f \in C_0(\mathbb{R}^n)$. Let $V \subset \mathbb{R}^n$ be open and bounded. By Lemma 1.63 there exists a sequence $f_j \in C_0(\mathbb{R}^n)$ s.t.

$$0 \le f_1(x) \le f_2(x) \le \cdots \le f_j(x) \to \chi_V(x)$$

for all $x \in \mathbb{R}^n$. By the Monotone Convergence Theorem

$$\begin{aligned} \mu_1(V) &= \lim_{j\to\infty} \int f_j\,d\mu_1 \\ &= \lim_{j\to\infty} \Lambda(f_j) \\ &= \lim_{j\to\infty} \int f_j\,d\mu \\ &= \mu(V). \end{aligned}$$

Because $\operatorname{Bor}(\mathbb{R}^n)$ is a σ-algebra spanned by open and bounded sets, then $\mu_1 = \mu$ (in $\operatorname{Bor}(\mathbb{R}^n)$).

□

6.10 Proof of Theorem 1.67

Proof. (a) Let $K \subset \mathbb{R}^n$ be compact and $V \subset \mathbb{R}^n$ be open s.t. $K \subset V$. Choose by part (a) of Lemma 1.63 a function $f \in C_0(\mathbb{R}^n)$ with $\chi_K \le f \le 1$ and $\operatorname{supp}(f) \subset V$. Then

$$\begin{aligned} \mu(V) &\ge \int_{\mathbb{R}^n} f\,d\mu \\ &= \lim_{k\to\infty} \int_{\mathbb{R}^n} f\,d\mu_k \\ &\ge \limsup_{k\to\infty} \mu_k(K). \end{aligned}$$

Because this holds for all open $V \supset K$, the claim in part (a) is proven.

(b) If $V \subset \mathbb{R}^n$ is open, then let $K \subset V$ compact. In the same way as above we obtain

$$\mu(K) \le \liminf_{k\to\infty} \mu_k(V).$$

Because μ is a Radon measure,

$$\mu(V) = \sup\{\mu(K) \colon K \subset V \text{ compact}\},$$

and part (b) is proven.

□

6.11 Proof of Theorem 1.69

For the proof we need the following auxiliary result.

Lemma 6.12. *The norm space $\big(C_0(\mathbb{R}^n), \|\cdot\|\big)$, where $\|f\| = \sup\{|f(x)| \colon x \in \mathbb{R}^n\}$, is separable, i.e. there is a countable dense set $\mathcal{F} = \{f_j\}_{j=1}^{\infty} \subset C_0(\mathbb{R}^n)$. In other words, if $f \in C_0(\mathbb{R}^n)$ and $\varepsilon > 0$, then $\|f - f_j\| < \varepsilon$ for some $f_j \in \mathcal{F}$.*

Proof. (Exerc.)

Proof of Theorem 1.69. Suppose first that

$$\sup_k \mu_k(\mathbb{R}^n) = A < \infty. \tag{6.13}$$

Let $\{f_j\}_{j=1}^{\infty}$ be a dense set in $C_0(\mathbb{R}^n)$. It follows from the assumption (6.13) that

$$\Big\{\int_{\mathbb{R}^n} f_1 \, d\mu_k \colon k \in \mathbb{N}\Big\}$$

is a bounded subset of $\mathbb{R}$, and hence there is a subsequence (μ_k^1) of (μ_k) s.t.

$$\int_{\mathbb{R}^n} f_1 \, d\mu_k^1 \xrightarrow{k\to\infty} a_1$$

for some $a_1 \in \mathbb{R}$. Choose inductively for all $j \ge 2$ a subsequence $\{\mu_k^j\}$ of the sequence $\{\mu_k^{j-1}\}$ s.t.

$$\int_{\mathbb{R}^n} f_j \, d\mu_k^j \xrightarrow{k\to\infty} a_j$$

for some $a_j \in \mathbb{R}$. Then the diagonal sequence $\{\mu_k^k\}_{k=1}^{\infty}$ satisfies

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} f_j \, d\mu_k^k = a_j \tag{6.14}$$

for all $j \ge 1$. Let L be the vector space spanned by the functions f_j,

$$L = \Big\{g = \sum_{j=1}^{m} \lambda_j f_j \colon \lambda_j \in \mathbb{R},\ m \in \mathbb{N}\Big\}.$$

Set

$$\Lambda(g) = \sum_j \lambda_j a_j,$$

when

$$g = \sum_j \lambda_j f_j.$$

Then by (6.14) we see that

$$\Lambda(g) = \lim_{k\to\infty} \int_{\mathbb{R}^n} g \, d\mu_k^k$$

for all $g \in L$. In particular, Λ is well defined (i.e. $\Lambda(g)$ is independent of the particular choice of the linear combination for g), positive and linear functional in L. Moreover, it follows from (6.13) that

$$|\Lambda(g)| \le A\,\|g\| \tag{6.15}$$

for all $g \in L$. If $f \in C_0(\mathbb{R}^n)$ is arbitrary, then choose a sequence (h_j), $h_j \in L$, s.t.

$$\|f - h_j\| \xrightarrow{j\to\infty} 0,$$

and set

$$\Lambda(f) = \lim_{j\to\infty} \Lambda(h_j).$$

Then it follows from (6.15) that Λ is well defined in $C_0(\mathbb{R}^n)$ ($\Lambda(f)$ independent of the choice of the sequence (h_j)) and (6.15) holds for all $g \in C_0(\mathbb{R}^n)$. Furthermore, Λ is a positive linear functional in $C_0(\mathbb{R}^n)$. In fact, if $f \ge 0$ and $\|f - h_j\| \to 0$, then $\liminf_{j\to\infty}(\min h_j) \ge 0$, and hence

$$\Lambda(f) = \lim_{j\to\infty} \lim_{k\to\infty} \int_{\mathbb{R}^n} h_j \, d\mu_k^k \ge 0,$$

because $\int h_j \, d\mu_k^k \ge A \min(0, \min h_j)$. By Riesz' representation theorem there exists a Radon-measure μ s.t.

$$\Lambda(f) = \int_{\mathbb{R}^n} f \, d\mu$$

for all $f \in C_0(\mathbb{R}^n)$. We prove next that $\mu_k^k \rightharpoonup \mu$. Let $\varepsilon > 0$. For $f \in C_0(\mathbb{R}^n)$, we choose $g \in L$ such that $\|f - g\| \le \frac{\varepsilon}{2A}$. Then for large values of k

$$\begin{aligned}\Big|\Lambda(f) - \int_{\mathbb{R}^n} f \, d\mu_k^k\Big| &\le |\Lambda(f-g)| + \underbrace{\Big|\Lambda(g) - \int_{\mathbb{R}^n} g \, d\mu_k^k\Big|}_{\le\varepsilon} + \Big|\int_{\mathbb{R}^n} (g - f) \, d\mu_k^k\Big| \\ &\le A\|f-g\| + \varepsilon + A\|f-g\| \\ &\le 2\varepsilon.\end{aligned}$$

Therefore $\mu_k^k \rightharpoonup \mu$. Finally, we give up the hypothesis (6.13). From the assumption that

$$\sup_k \mu_k(K) < \infty$$

for all compact $K \subset \mathbb{R}^n$ and the above argument there follows that for every $m \in \mathbb{N}$ there exists a subsequence (μ_k^m) of (μ_k) s.t. $\{\mu_k^m : k \in \mathbb{N}\} \subset \{\mu_k^{m-1} : k \in \mathbb{N}\}$ and

$$\mu_k^m \llcorner B(0,m) \rightharpoonup \nu^m,$$

where ν^m is a Radon-measure with $\operatorname{supp}(\nu^m) \subset \bar{B}(0,m)$. Then for the diagonal sequence μ_k^k there holds

$$\mu_k^k \llcorner B(0,m) \rightharpoonup \nu^m$$

for all $m \in \mathbb{N}$. Thus $\nu^m \llcorner B(0,\ell) = \nu^\ell$, when $\ell \leq m$. Therefore we may define a Radon-measure μ in $\mathbb{R}^n$ setting

$$\mu(E) = \nu^1\big(E \cap B(0,1)\big) + \sum_{m \geq 2} \nu^m\Big(E \cap \big(B(0,m) \setminus B(0,m-1)\big)\Big), \quad E \in \operatorname{Bor}(R^n).$$

Because $\operatorname{supp}(f) \subset B(0,m_0)$, it follows that

$$\int_{\mathbb{R}^n} f \, d\mu = \int_{\mathbb{R}^n} f \, d\nu^{m_0} = \lim_{k \to \infty} \int_{\mathbb{R}^n} f \, d\mu_k^k,$$

and hence $\mu_k^k \rightharpoonup \mu$. □