FOURIER ANALYSIS. (fall 2016)

SKETCHES OF SOLUTIONS FOR THE REVIEW PROBLEM SET

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Exercise 1. Assume that $f, f_k \in L^2(-\pi,\pi)$ for $k \geq 1$ and $\lim_{k\to\infty} \|f_k - f\|_{L^2(-\pi,\pi)} = 0$. Show that for each $n \in \mathbb{Z}$ it holds that $\widehat{f_k}(n) \to \widehat{f}(n)$ as as $k \to \infty$.

Solution 1. Plancherel's formula states that for $g \in L^2(-\pi,\pi)$

$$\frac{1}{2\pi}\|g\|^2_{L^2(-\pi,\pi)} = \sum_{n=-\infty}^{\infty} |\widehat{g}(n)|^2.$$

The convergence now follows from

$$|\widehat{f_k}(n) - \widehat{f}(n)|^2 \leq \sum_{n=\infty}^{\infty} |\widehat{f_k}(n) - \widehat{f}(n)|^2.$$

Exercise 2. Let f be a 2π-periodic function such that $f(x) = 0$ for $x \in [-\pi, 0)$ and $f(x) = 1/(1+x)$ for $x \in [0,\pi)$. Show that the Fourier series of f does not converge absolutely.

Solution 2. If we assume that the Fourier series converges absolutely, then we see that the series

$$\sum_{n=-\infty}^{\infty} \widehat{f}(n)e^i nx$$

converges uniformly to f. But then f is a uniform limit of continuous functions, so it must be continuous. But we see that f is not continuous at 0, so the Fourier series cannot converge absolutely.

Exercise 3. Let $f : [-\pi,\pi) \to \mathbb{R}$ be a real-valued 2π-periodic function such that all the Fourier coefficients of f are real-valued. Show that f is even, i.e. that $f(x) = f(-x)$ for (almost) all $x \in \mathbb{R}$.

Solution 3. Compute the Fourier coefficients of f.

$$\widehat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx = \frac{1}{2\pi}\left(\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx - i\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx\right)$$

As we assume that the Fourier coefficients are real and f is real-valued, the rightmost integral must vanish for every n. From this we get that $\widehat{f}(n) = \widehat{f}(-n)$ as cosine is an even function. This implies that f is even.

Exercise 4. Assume that if $f, g \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ and denote their convolution by $h := f * g$.

(i) Show that $h \in L^\infty(\mathbb{R}^d)$.

(ii) Is it always true that $\widehat{h} \in L^{2016}(\mathbb{R}^d)$?

Solution 4. **(i)** We use the Cauchy-Schwartz inequality in $L^2(\mathbb{R}^d)$ to compute that for any $x \in \mathbb{R}^d$ we have

$$|h(x)| = \left| \int_{\mathbb{R}^d} f(y) g(x-y)\, dy \right| \leq \|f\|_{L^2(\mathbb{R}^d)} \|g\|_{L^2(\mathbb{R}^d)} < \infty.$$

Therefore $h \in L^\infty(\mathbb{R}^d)$.

(ii) We know from Fubini's theorem that the convolution of two L^1-functions is L^1. This means that $h \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. If we set $p = 2016/2015$, and denote $E = \{x \in \mathbb{R}^d : |h(x)| \leq 1\}$, then

$$\int_{\mathbb{R}^d} |h(x)|^p\, dx = \int_E |h(x)|^p\, dx + \int_{\mathbb{R}^d \setminus E} |h(x)|^p\, dx \leq \int_E |h(x)|\, dx + m_d(\mathbb{R}^d \setminus E) \|h\|^p_{L^\infty(\mathbb{R}^d)} < \infty.$$

Now we see that $h \in L^{\frac{2016}{2015}}(\mathbb{R}^d)$ and we know by interpolation that the Fourier transform takes $L^{\frac{2016}{2015}}(\mathbb{R}^d)$ to $L^{2016}(\mathbb{R}^d)$. This shows that $\widehat{h} \in L^{2016}(\mathbb{R}^d)$

Exercise 5. Denote $f_r(x) = e^{-rx^2}$ for $r > 0$.. Given $r_1, r_2 > 0$, determine the convolution $f_{r_1} * f_{r_2}$.

Solution 5. We will determine the Fourier transform of the convolution. Recall that

$$\widehat{f_r}(\xi) = \sqrt{\frac{\pi}{r}} e^{-\frac{x^2}{4r}}.$$

and that $\widehat{g * h}(\xi) = \widehat{g}(\xi)\widehat{h}(\xi)$. This gives us

$$\widehat{f_{r_1} * f_{r_2}}(\xi) = \widehat{f}_{r_1}(\xi)\widehat{f}_{r_2}(\xi) = \sqrt{\frac{\pi^2}{r_1 r_2}} e^{-x^2\left(\frac{1}{4r_1} + \frac{1}{4r_2}\right)}.$$

From this we can solve

$$f_{r_1} * f_{r_2}(x) = \sqrt{\frac{\pi}{r_1 + r_2}} e^{-x^2 \frac{r_1 r_2}{r_1 + r_2}}.$$

Exercise 6. Let $f \in L^1(\mathbb{R}^d) \cap C(\mathbb{R}^d)$ and assume that $\widehat{f}(\xi) \geq 0$ for all $\xi \in \mathbb{R}^d$. Prove that then $\widehat{f} \in L^1(\mathbb{R}^d)$.

Solution 6. Let $\phi \in C_c^\infty(\mathbb{R}^d)$ have $0 \leq \phi(x) \leq 1$ for all $x \in \mathbb{R}^d$, $\phi(x) = 1$ for $|x| \leq 1$ and $\phi(x) = 0$ for $|x| \geq 2$. Define $\phi_n(x) := \phi(x/n)$.

Now for any n

$$\int_{B(0,n)} |\widehat{f}(\xi)|\, d\xi \le \int_{\mathbb{R}^d} \widehat{f}(\xi)\phi_n(\xi)\, d\xi = \int_{\mathbb{R}^d} f(x)\widehat{\phi}_n(x)\, dx = \int_{\mathbb{R}^d} f(x)n^d\widehat{\phi}(nx)\, dx \to Cf(0) < \infty,$$

where the limit is because $n^d\widehat{\phi}(nx)$ gives a sequence of good kernels up to constant factors. As the sequence dominating $\int_{B(0,n)} |\widehat{f}(\xi)|\, d\xi$ has a finite limit, we get that $\widehat{f} \in L^1(\mathbb{R}^d)$.

Exercise 7. What of the following claims are true? Give justification for your answer (you may use all the results of the lectures).

a) Distribution $|a|\delta_a$ tends to zero in the space $\mathcal{S}'(\mathbb{R}^d)$ as $|a| \to \infty$.

b) If $f : \mathbb{R}^d \to \mathbb{R}$ is bounded and compactly supported, then $\widehat{f} \in L^2(\mathbb{R})$.

c) If the distribution $T \in \mathcal{S}'(\mathbb{R}^d)$ is supported at one point $a \in \mathbb{R}^d$, then $\widehat{T}$ is a bounded function.

d) If $g \in \mathcal{S}(\mathbb{R})$ and $g(0) = 0$, then in the metric of $\mathcal{S}(\mathbb{R})$ one has that $g(nx) \to 0$ as $n \to \infty$.

Solution 7. a) This is true. For any $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$ we have $(1 + |x|^2)|\varphi(x)| \le p_1(\varphi)$, so

$$|\langle |a|\delta_a, \varphi\rangle| = |a||\varphi(a)| \le \frac{|a|p_1(\varphi}{1 + |a|^2} \to 0$$

as $|a| \to \infty$.

b) If f is not measurable, then $\widehat{f}$ is not defined, making the claim untrue in that case.

If we assume f to be measurable, then this is true. In this case, the square of L^2-norm of f is bounded by $m_d(K)\|f\|^2_{L^\infty(\mathbb{R}^d)}$, where K is the support of f. Then $f \in L^2(\mathbb{R}^d)$, so $\widehat{f} \in L^2(\mathbb{R}^d)$.

c) This is false. Take $T = \frac{\partial}{\partial x_1}\delta_0$. Then T is supported in a single point, but $\widehat{T} = ix_1$ is not a bounded function.

d) This is false. Let $g \in \mathcal{S}(\mathbb{R})$ have $g(0) = 0$ and $g'(0) = 1$. Then if we denote $g_n(x) = g(nx)$, we have

$$p_1(g_n) \ge g_n'(0) = n \to \infty.$$

As g_n is not a bounded sequence in the seminorm p_1, it cannot converge in that seminorm and hence not in $\mathcal{S}(\mathbb{R})$.

Exercise 8. Let $\lambda \in \mathcal{S}'(\mathbb{R})$ and assume that ϕ_n are smooth compactly supported functions such that $\phi_n \to \lambda$ as $n \to \infty$ in the sense of distribution. Assume that we know that $\lim_{n\to\infty} \phi_n(x) = 0$ for every $x \in \mathbb{R}$. Does it follow that $\lambda = 0$?

Solution 8. It does not follow that $\lambda = 0$. Let ϕ be a smooth compactly supported function with $\phi(0) = 0$ and $\int_{\mathbb{R}} \phi(x)\, dx = 1$. Set $\phi_n(x) = n\phi(nx)$. Now from previous exercises we know that $\phi_n \to \delta_0$ in the sense of distributions, and we see that $\lim_{n\to\infty} \phi_n(x) = 0$ for any x.