FOURIER ANALYSIS. (Fall 2016)

REVIEW EXERCISES (Wed 7.12, 9-12 in room C123)

These exercises are not part of the course, and you will not get points from them. They are just to help your preparation for the test. Also, note that they are pretty randomly chosen and cover only small portion of the subject – they are just to help your own preparation for the test.

1. Assume that $f, f_k \in L^2(-\pi,\pi)$ for $k \geq 1$ and $\lim_{k\to\infty} \|f_k - f\|_{L^2(-\pi,\pi)} = 0$. Show that for each $n \in \mathbf{Z}$ it holds that $\widehat{f_k}(n) \to \widehat{f}(n)$ as as $k \to \infty$.

2. Let f be a 2π-periodic function such that $f(x) = 0$ for $x \in [-\pi, 0)$ and $f(x) = 1/(1+x)$ for $x \in [0,\pi)$. Show that the Fourier series of f does not converge absolutely.

3. Let $f : [-\pi,\pi) \to \mathbf{R}$ be a real-valued 2π-periodic function such that all the Fourier coefficients of f are real-valued. Show that f is even, i.e. that $f(x) = f(-x)$ for (almost) all $x \in \mathbf{R}$.

4. Assume that if $f, g \in L^1(\mathbf{R}^d) \cap L^2(\mathbf{R}^d)$ and denote their convolution by $h := f * g$.

(i) Show that $h \in L^\infty(\mathbf{R}^d)$.

(ii) Is it always true that $\widehat{h} \in L^{2016}(\mathbf{R}^d)$?

5. Denote $f_r(x) = e^{-rx^2}$ for $r > 0$.. Given $r_1, r_2 > 0$, determine the convolution $f_{r_1} * f_{r_2}$.

[Hint: Use the results of the lectures and compute on the Fourier side.]

6. Let $f \in L^1(\mathbf{R}^d) \cap C(\mathbf{R}^d)$ and assume that $\widehat{f}(\xi) \geq 0$ for all $\xi \in \mathbf{R}^d$. Prove that then $\widehat{f} \in L^1(\mathbf{R}^d)$.

[Suggestion: Pick a suitable test function $\phi \in \mathcal{S}(\mathbf{R}^d)$ with $\phi \geq 0$. Denote $\phi_n(x) = \phi(nx)$ and apply the identity of Thm. 9.7 on f and ϕ_n. Let $n \to \infty$.]

7. What of the following claims are true? Give justification for your answer (you may use all the results of the lectures).

a) Distribution $|a|\delta_a$ tends to zero in the space $\mathcal{S}'(\mathbf{R}^d)$ as $|a| \to \infty$.

b) If $f : \mathbf{R}^d \to \mathbf{R}$ is bounded and compactly supported, then $\widehat{f} \in L^2(\mathbf{R})$.

c) If the distribution $T \in \mathcal{S}'(\mathbf{R}^d)$ is supported at one point $a \in \mathbf{R}^d$, then $\widehat{T}$ is a bounded function.

d) If $g \in \mathcal{S}(\mathbf{R})$ and $g(0) = 0$, then in the metric of $\mathcal{S}(\mathbf{R})$ one has that $g(nx) \to 0$ as $n \to \infty$.

8. Let $\lambda \in \mathcal{S}'(\mathbf{R})$ and assume that ϕ_n are smooth compactly supported functions such that $\phi_n \to \lambda$ as $n \to \infty$ in the sense of distribution. Assume that we know that $\lim_{n\to\infty} \phi_n(x) = 0$ for every $x \in \mathbf{R}$. Does it follow that $\lambda = 0$?