FOURIER ANALYSIS. (fall 2016)

MODEL SOLUTIONS FOR SET 9

Exercise 1. Determine the fundamental solution of Laplacian in 1-dimension, i.e. find $E \in \mathcal{S}'(\mathbb{R})$ so that $\left(\frac{d}{dx}\right)^2 E = \delta_0$.

Solution 1. Recall the Heaviside step function

$$H(x) = \chi_{[0,\infty)}.$$

As distribution, we know that

$$\frac{d}{dx} H(x) = \delta_0.$$

It now suffices to find an absolutely continuous function whose derivative is Heaviside function plus some constant: this is an ordinary differential equation and one possible solution is

$$E(x) = |x|/2.$$

Exercise 2. Use the Poisson summation formula to prove

$$\sum_{n \in \mathbb{Z}} \frac{1}{1+n^2} = \pi \frac{1+e^{-2\pi}}{1-e^{-2\pi}}$$

Solution 2. Let us recall from the previous set of exercises that if $f(x) = e^{-|x|}$, then

$$\widehat{f}(\xi) = \frac{2}{1+\xi^2}.$$

By the Fourier inversion formula we have also that

$$\mathcal{F}\left(\widehat{f}\right)(x) = 2\pi e^{-|x|}.$$

We want to apply the Poisson summation formula on the function $\widehat{f}$. We must check that

$$|\widehat{f}(\xi)| \leq C(1+|\xi|)^{-1-\epsilon} \quad \text{and} \quad \left|\mathcal{F}\left(\widehat{f}\right)(x)\right| \leq C(1+|\xi|)^{-1-\epsilon}$$

for some constants $C, \epsilon > 0$. The second condition holds for every $\epsilon > 0$ since the exponential function grows faster than any polynomial. The first condition works for $\epsilon = 1$ because of the estimate

$$|\widehat{f}(\xi)| = \frac{2}{1+\xi^2} \leq \frac{4}{(1+|\xi|)^2}.$$

Thus we can apply the Poisson summation formula to get that

$$2\sum_{n \in \mathbb{Z}} \frac{1}{1+n^2} = \sum_{n \in \mathbb{Z}} \widehat{f}(n) = \sum_{n \in \mathbb{Z}} \mathcal{F}\left(\widehat{f}\right)(2\pi n) = \sum_{n \in \mathbb{Z}} 2\pi e^{-|n|}.$$

The sum on the right hand side is a combination of two geometric sums and thus easy to compute. In the end we get that

$$\sum_{n\in\mathbb{Z}} \frac{1}{1+n^2} = \pi \frac{1+e^{-2\pi}}{1-e^{-2\pi}}.$$

Exercise 3. **(i)** Suppose $A : \mathbb{R}^d \to \mathbb{R}^d$ is an invertible linear map (we denote by A also its matrix). If $f \in L^1(\mathbb{R}^d)$, define $g(x) = f(Ax)$. Show that

$$\widehat{g}(\xi) = \frac{1}{|\det(A)|}\widehat{f}((A^{-1})^T\xi),$$

where $(A^{-1})^T$ is the transpose of the inverse of A.

(ii) A function $f \in L^1(\mathbb{R}^d)$ is radial if $f(x)$ depends only on $|x|$. Use (i) to show that for a radial function, the Fourier transform is radial.

(iii) Show that the result in (ii) holds also for every radial $f \in L^2(\mathbb{R}^d)$ in a sense that the Fourier transform $\widehat{f} \in L^2(\mathbb{R}^d)$ has a radial representative.

Solution 3. **(i)** We compute via the change of variables formula that

$$\begin{aligned}
\widehat{g}(\xi) &= \int_{\mathbb{R}^d} e^{-i\xi\cdot x} f(Ax)dx \\
&= \frac{1}{|\det A|}\int_{\mathbb{R}^d} e^{-i\xi\cdot A^{-1}x} f(x)dx \\
&= \frac{1}{|\det A|}\int_{\mathbb{R}^d} e^{-i(A^{-1})^T\xi\cdot x} f(x)dx \\
&= \frac{1}{|\det A|}\widehat{f}((A^{-1})^T\xi).
\end{aligned}$$

We also used the property of the matrix transpose that $\xi \cdot A^{-1}x = (A^{-1})^T\xi \cdot x$.

(ii) Let f be a radial L^1 function. Then $f(Ax) = f(x)$. By the first part we get that

$$\widehat{f}(\xi) = \frac{1}{|\det A|}\widehat{f}((A^{-1})^T\xi) = \widehat{f}((A^{-1})^T\xi),$$

since $\det A = \pm 1$. Choosing $A = (B^{-1})^T$ for some other arbitrary rotation B gives that

$$\widehat{f}(\xi) = \widehat{f}(B\xi)$$

for every rotation B. Since $\widehat{f}$ is also continuous, it must be a radial function.

(iii) Let f be a radial L^2 function. Then $f_M = f \cdot \xi_{B(0,M)}$ is also radial and we have that

$$\widehat{f_M} \to \widehat{f}$$

in L^2. The functions $\widehat{f_M}$ are radial by (ii) so we now want to conclude that $\widehat{f}$ is also radial (so we want to prove that the L^2-limit of radial functions is radial).

!! Warning !! It is not enough to prove that $\widehat{f}(B\xi) = \widehat{f}(\xi)$ for every rotation B (as L^2-functions). The problem comes from the fact that this only proves that for every rotation B, the identity $\widehat{f}(B\xi) = \widehat{f}(\xi)$ holds pointwise almost everywhere. The set of zero measure in which this identity fails might depend on B! Thus it is not immediately obvious why there is **also** a radial representative for our function f in its equivalence class in L^2. Let us look for a different approach.

Our original proof is based on the Lebesgue set:

$$N = \{x \in \mathbb{R}^n : x \text{ is a Lebesgue point for } \widehat{f}\}.$$

Indeed, if $x, y \in N$ and $|x| = |y|$, then

$$\begin{aligned}
\widehat{f}(x) &= \lim_{r\to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} \widehat{f}(z)dz \\
&= \lim_{r\to 0} \lim_{M\to\infty} \frac{1}{|B(x,r)|} \int_{B(x,r)} \widehat{f_M}(z)dz \\
&= \lim_{r\to 0} \lim_{M\to\infty} \frac{1}{|B(x,r)|} \int_{B(y,r)} \widehat{f_M}(z)dz \\
&= \lim_{r\to 0} \frac{1}{|B(y,r)|} \int_{B(y,r)} \widehat{f}(z)dz \\
&= \widehat{f}(y).
\end{aligned}$$

Thus f is radial in the Lebesgue set. The complement of the Lebesgue set is of measure zero, so we can redefine f in the complement so that it is radial everywhere.

Another proof for the same fact is to choose a subsequence of $\widehat{f_M}$ that converges to $\widehat{f}$ pointwise almost everywhere. This is possible as proven in the real analysis course. Since $\widehat{f}$ is almost everywhere a pointwise limit of radial functions, it must have a radial representative in L^2.

Exercise 4. Show that if E is a fundamental solution of the differential operator (with constant coefficients) $P(\partial)$, then $E + H$ is also a fundamental solution, if $H \in \mathcal{S}'(\mathbb{R}^d)$ satisfies $P(\partial)H = 0$.Verify that actually all fundamental solutions of P are obtained by this manner.

Solution 4. As the considered differential operator is linear, we see that for any $\varphi \in \mathcal{S}(\mathbb{R}^d)$

$$\langle P(\partial)(E + H), \varphi\rangle = \langle P(\partial)E, \varphi\rangle + \langle P(\partial)H, \varphi\rangle = \langle \delta_0, \varphi\rangle + \langle 0, \varphi\rangle = \langle \delta_0, \varphi\rangle.$$

Next assume that E_1, E_2 are two fundamental solutions. Then $E_2 = E_1 + (E_2 - E_1)$ and we have for any $\varphi \in \mathcal{S}(\mathbb{R}^d)$

$$\langle P(\partial)(E_2 - E_1), \varphi\rangle = \langle P(\partial)E_2, \varphi\rangle - \langle P(\partial)E_1, \varphi\rangle = \langle \delta_0, \varphi\rangle - \langle \delta_0, \varphi\rangle = 0.$$

Therefore all fundamental solutions can be obtained from one by adding $H \in \mathcal{S}'(\mathbb{R}^d)$ that satisfies $P(\partial)H = 0$.

Exercise 5. Recall that we proved that at the function (an example of Weierstrass functions)

$$f(x) := \sum_{n=1}^{\infty} 2^{-n/2} \cos(2^n x)$$

is not differentiable at any point. Show in any case that in the sense of distributions we have

$$f'(x) = -\sum_{n=1}^{\infty} 2^{n/2} \sin(2^n x) \qquad !$$

Solution 5. Define

$$f_N = \sum_{n=1}^{N} 2^{-n/2} \cos(2^n x).$$

Then the f_N are continuous functions and $f_N \to f$ uniformly as $n \to \infty$. The uniform convergence follows from

$$|f(x) - f_N(x)| \leq \sum_{n=N+1}^{\infty} 2^{-n/2} \to 0.$$

Thus $f_N \to f$ in the sense of distributions, which implies that $f'_N \to f'$ in the sense of distributions. Thus

$$f'_N(x) = -\sum_{n=1}^{N} i2^n 2^{-n/2} sin^{2^n x} = -\sum_{n=1}^{N} i2^{n/2} sin^{2^n x} \to f'$$

in the sense of distributions, which is what we wanted to prove.

Exercise 6. Let $A = \{(x, y) : x > 0, y > 0\} \cup \{(x, y) : x < 0, y < 0\} \subset \mathbb{R}^2$.

Show that the characteristic function χ_A is a fundamental solution for the differential operator $P_1(\partial) = \frac{1}{2}\partial_1\partial_2$.

Solution 6. We need to check that in the sense of distributions the following identity holds:

$$P_1(\partial)\chi_A = \delta_0.$$

This is just a simple calculation as follows:

$$\begin{aligned}
\langle P_1(\partial)\chi_A, g\rangle &= \left\langle \chi_A, \frac{1}{2}g_{xy}\right\rangle \\
&= \iint_A g_{xy}(x,y)dxdy \\
&= \frac{1}{2}\int_0^\infty \int_0^\infty g_{xy}(x,y)dxdy + \frac{1}{2}\int_{-\infty}^0 \int_{-\infty}^0 g_{xy}(x,y)dxdy \\
&= \frac{1}{2}\int_0^\infty -g_y(0,y)dy + \frac{1}{2}\int_{-\infty}^0 g_y(0,y)dy \\
&= \frac{g(0,0)}{2} + \frac{g(0,0)}{2} \\
&= \langle \delta_0, g\rangle .
\end{aligned}$$

Exercise 7. (i) If $0 < \gamma < d$, show that the function

$$f_\gamma(x) = \frac{1}{|x|^\gamma}, \qquad x \in \mathbb{R}^d \setminus \{0\},$$

determines a tempered distribution, by writing it as a sum of two functions, one belonging to $L^1(\mathbb{R}^d)$ and the other to $L^p(\mathbb{R}^d)$ for a suitable $p > 1$. If $\gamma > d/2$, show that one can choose $p = 2$.

(ii) Prove that

$$\widehat{f_\gamma}(\xi) = c(d,\gamma)\frac{1}{|x|^{d-\gamma}}$$

for some constant $c(d,\gamma)$.

Solution 7. (i) If $0 < \gamma < d$, we write that

$$f_\gamma(x) = \chi_{\{|x|<1\}}\frac{1}{|x|^\gamma} + \chi_{\{|x|\geq 1\}}\frac{1}{|x|^\gamma}.$$

The first part is in L^1 since by a spherical change of variables we find that

$$\int_{|x|<1} \frac{1}{|x|^\gamma}dx = C\int_0^1 \frac{1}{r^\gamma}r^{d-1}dr,$$

and since the exponent $d-\gamma-1$ is greater than -1, this integral is finite. For the second part to be in L^p, the integral

$$\int_{|x|\geq 1} \frac{1}{|x|^{p\gamma}}dx = C\int_1^\infty \frac{1}{r^{p\gamma}}r^{d-1}dr$$

must be finite. Thus we get the inequality $d - p\gamma - 1 < -1$, which is true when $p > d/\gamma$. If $\gamma > d/2$, we see from this that one may choose $p = 2$. Thus in general f_γ is a sum of an L^1-function and an L^p-function, and hence defines a tempered distribution on $\mathbb{R}^d$.

(ii) Case $\gamma > d/2$. In this case f_γ is a sum of an L^1-function and a L^2-function. Thus we find that the Fourier transform $\widehat{f_\gamma}$ is also a function (it is a sum of an L^∞-function and a L^2-function). Note that the function f_γ has the property that

$$f_\gamma(tx) = t^{-\gamma} f_\gamma(x) \quad \text{for every } t > 0 \text{ and every } x.$$

Let $g \in \mathcal{S}(\mathbb{R}^d)$. Then

$$\begin{aligned}
\left\langle \widehat{f_\gamma}(x/t), g(x) \right\rangle &= \left\langle \widehat{f_\gamma}(x), t^d g(tx) \right\rangle \\
&= \left\langle \widehat{f_\gamma}(x), t^d g(tx) \right\rangle \\
&= \left\langle f_\gamma(x), \widehat{t^d g(tx)} \right\rangle \\
&= \langle f_\gamma(x), \widehat{g}(x/t) \rangle \\
&= \langle t^d f_\gamma(tx), \widehat{g}(x) \rangle \\
&= t^{d-\gamma} \langle f_\gamma(x), \widehat{g}(x) \rangle \\
&= \left\langle t^{d-\gamma} \widehat{f_\gamma}(x), g(x) \right\rangle .
\end{aligned}$$

Thus $\widehat{f_\gamma}(x/t) = t^{d-\gamma} \widehat{f_\gamma}(x)$ in the distributional sense. The above formula can also be generalized to functions g in $L^1 \cap L^2$ by approximation. We want to conclude that there is a representative of the function $\widehat{f_\gamma}$ (which is defined almost everywhere) that also satisfies the identity

$$\widehat{f_\gamma}(x/t) = t^{d-\gamma} \widehat{f_\gamma}(x) \tag{1}$$

at every point $x \in \mathbb{R}^n$. For this we again use the Lebesgue set

$$N = \{x \in \mathbb{R}^n : x \text{ is a Lebesgue point for } \widehat{f_\gamma}\}.$$

If $x \in N$ and $x/t \in N$, then a computation similar to the one in Exercise 2(iii) gives that

$$\begin{aligned}
\widehat{f_\gamma}(x/t) &= \lim_{r \to 0} \frac{1}{|B(x/t, r)|} \int_{B(x/t,r)} \widehat{f_\gamma}(z) dz \\
&= \lim_{r \to 0} \frac{1}{|B(x/t, r)|} \int_{B(x,tr)} t^{-d} \widehat{f_\gamma}(z/t) dz \\
&= \lim_{r \to 0} \frac{1}{|B(x/t, r)|} \left\langle t^{-d} \widehat{f_\gamma}(z/t), \chi_{B(x,tr)}(z) \right\rangle \\
&= \lim_{r \to 0} \frac{t^d}{|B(x, tr)|} \left\langle t^{-\gamma} \widehat{f_\gamma}(z), \chi_{B(x,tr)}(z) \right\rangle \\
&= \lim_{r \to 0} \frac{t^{d-\gamma}}{|B(x, tr)|} \int_{B(x,tr)} \widehat{f_\gamma}(z) dz \\
&= t^{d-\gamma} \widehat{f_\gamma}(x).
\end{aligned}$$

The computation is valid since $\chi_{B(x,tr)}(z) \in L^1 \cap L^2$. Since the Lebesgue set N contains almost every point in $\mathbb{R}^d$, we can redefine f outside of N so that it satisfies the identity

$\widehat{f_\gamma}(x/t) = t^{d-\gamma}\widehat{f_\gamma}(x)$ everywhere. Now we use this formula to compute that for every ξ we have

$$\widehat{f_\gamma}(\xi) = \widehat{f_\gamma}(\xi/|\xi|)\frac{1}{|\xi|^{d-\gamma}}.$$

The point $\xi/|\xi|$ is on the unit sphere and by Exercise 2 our function $\widehat{f_\gamma}$ is radial (by combining the arguments we can actually choose a representative that is both radial and satisfies the identity (1)). Thus $\widehat{f_\gamma}(\xi/|\xi|)$ does not depend on the choice of ξ. Hence we can denote it by a constant $c(\gamma, d)$ and we get that

$$\widehat{f_\gamma}(\xi) = \frac{c(\gamma, d)}{|\xi|^{d-\gamma}}$$

as wanted. **Remark:** It was very important to choose the correct representative of f, as the unit sphere has zero measure in $\mathbb{R}^d$ and thus the value of $c(\gamma, \delta)$ would otherwise depend on the representative chosen, which would make no sense.

Case $\gamma < d/2$. In this case we have that $d - \gamma > d/2$. From the previous case it follows that

$$\widehat{f_{d-\gamma}}(\xi) = \frac{c(d-\gamma, d)}{|\xi|^{\gamma}} = c(d-\gamma, d)f_\gamma(\xi).$$

This Fourier transform is taken in the sense of distributions. We can take the inverse Fourier transform of both sides (again in the sense of distributions) to get that

$$f_{d-\gamma}(x) = c(d-\gamma, d)\mathcal{F}^{-1}f_\gamma(x) = c(d-\gamma, d)2\pi\widehat{f_\gamma}(x).$$

The constant $c(d-\gamma, d)$ cannot be zero, so we get that

$$\widehat{f_\gamma}(\xi) = \frac{c(\gamma, d)}{|\xi|^{d-\gamma}} \quad \text{where} \quad c(\gamma, d) = \frac{1}{c(d-\gamma, d)2\pi}$$

as wanted.

Case $\gamma = d/2$. Let us investigate what happens when $\gamma \to d/2$. By dominated convergence we have that

$$\langle f_\gamma, g\rangle = \int_{\mathbb{R}^d} \frac{g(x)}{|x|^\gamma}dx \to \int_{\mathbb{R}^d} \frac{g(x)}{|x|^{d/2}} = \langle f_{d/2}, g\rangle.$$

This shows that $f_\gamma \to f_{d/2}$ in $\mathcal{S}'(\mathbb{R}^d)$. Thus we must also have that $\widehat{f_\gamma} \to \widehat{f_{d/2}}$ in $\mathcal{S}'(\mathbb{R}^d)$ as $\gamma \to d/2$, since we already know that the distribution $\widehat{f_{d/2}}$ is well-defined. Now let us choose a subsequence γ_k so that the numbers $c(\gamma_k, d)$ converge either to a real number c or to $\pm\infty$. We can compute again by dominated convergence that

$$\begin{aligned}\lim_{k\to\infty}\left\langle \widehat{f_{\gamma_k}}, g\right\rangle &= \lim_{k\to\infty}\int_{\mathbb{R}^d} \frac{c(\gamma_k, d)}{|x|^{d-\gamma_k}}g(x)dx \\ &= \left(\lim_{k\to\infty} c(\gamma_k, d)\right)\int_{\mathbb{R}^d} \frac{1}{|x|^{d/2}}g(x)dx.\end{aligned}$$

We know that the limit on the right hand side cannot be $\pm\infty$ since the $\widehat{f_{\gamma_k}}$ must have a distributional limit (which is $\widehat{f_{d/2}}$). Thus the $c(\gamma_k, d)$ must converge to a real number c. This gives that

$$\left\langle \widehat{f_{d/2}}, g \right\rangle = \lim_{k\to\infty} \left\langle \widehat{f_{\gamma_k}}, g \right\rangle = \left\langle c/|\xi|^{d/2}, g \right\rangle.$$

It follows that $\widehat{f_{d/2}} = c/|\xi|^{d/2}$ as wanted.

Remark. It is in fact possible to determine the constants $c(\gamma, d)$ exactly with help of the function $e^{-|x|^2}$, whose Fourier transform is easy enough to find. The value turns out to be

$$c(\gamma, d) = \frac{\pi^{d/2} 2^{d-\gamma} \Gamma\left(\frac{d-\gamma}{2}\right)}{\Gamma(\gamma/2)},$$

where Γ is the gamma function

$$\Gamma(s) := \int_0^\infty t^{s-1} e^- s \, ds.$$

Exercise 8*. Try to figure out how might a fundamental solution of Δ^2 look in $\mathbb{R}^3$!

Solution 8*. (sketch) We try to find the distribution E with

$$\Delta^2 E = \delta_0.$$

Suppose that E is such a fundamental solution. Taking the Fourier transform and recalling Exercise 5 from Set 6, we see that this is equivalent to

$$|\xi|^4 \widehat{E} = 1.$$

Here we run into a problem: the function $\xi \mapsto |\xi|^{-4}$ is not locally integrable and does not define a tempered distribution in an obvious way.

However, we can get past this problem by considering a suitable singular integral. Denote by λ the operator with

$$\langle \lambda, g \rangle := \lim_{\varepsilon\to 0+} \int_{|x|>\varepsilon} \frac{g(x) - g(0)}{|x|^4} \, dx,$$

if the limit exists. We claim that the limit exists for any $g \in \mathcal{S}(\mathbb{R}^3)$ and that $\lambda \in \mathcal{S}'(\mathbb{R}^3)$. We split the integral into two parts:

$$\int_{|x|>\varepsilon} \frac{g(x) - g(0)}{|x|^4} \, dx = \int_{|x|\geq 1} \frac{g(x) - g(0)}{|x|^4} \, dx + \int_{\varepsilon<|x|<1} \frac{g(x) - g(0)}{|x|^4} \, dx.$$

We can estimate the first part as

$$\left| \int_{|x|\geq 1} \frac{g(x)-g(0)}{|x|^4}\, dx \right| \leq \int_{|x|\geq 1} \frac{2p_0(g)}{|x|^4}\, dx = Dp_0(g)$$

where D is some independent constant. For the second part, we know that the second partial derivatives of g are bounded by $p_2(g)$, so we have the following Taylor approximation near 0:

$$g(x) = g(0) + \nabla g(0)\cdot x + |x|^2 A(x),$$

where A is a function that is bounded by $C_0 p_2(g)$ for some constant C_0 in the neighbourhood of zero. Then we can use the fact that $\nabla g(0)\cdot x$ is odd to estimate

$$\begin{aligned}\left| \int_{\varepsilon<|x|<1} \frac{g(x)-g(0)}{|x|^4}\, dx \right| &= \left| \int_{\varepsilon<|x|<1} \left(\frac{\nabla g(0)\cdot x}{|x|^4} + \frac{A(x)}{|x|^2} \right) dx \right| \\ &= \left| \int_{\varepsilon<|x|<1} \frac{A(x)}{|x|^2}\, dx \right| \to \left| \int_{|x|<1} \frac{A(x)}{|x|^2}\, dx \right| \\ &\leq \int_{|x|<1} \frac{C_0 p_2(g)}{|x|^2}\, dx = C_1 p_2(g)\end{aligned}$$

Here we used the fact that $x \mapsto |x|^{-2}$ is locally integrable. As λ is linear and bounded by the seminorm p_2, we see that $\lambda \in \mathcal{S}'(\mathbb{R}^3)$.

We observe that $\langle \lambda, |x|^4 g\rangle = \langle 1, g\rangle$, so now we need to find a tempered distribution E with $\widehat{E} = \lambda$. Recall from Exercise 4 of the previous set that $\langle \mathcal{F}^{-1}T, g\rangle = \langle T, \mathcal{F}^{-1}g\rangle$ for any $T \in \mathcal{S}'(\mathbb{R}^3)$. So we will now use the inversion formula:

$$\begin{aligned}\langle E, g\rangle &= \langle \mathcal{F}^{-1}\lambda, g\rangle = \langle \lambda, \mathcal{F}^{-1}g\rangle \\ &= \lim_{\varepsilon\to 0+} \int_{|\xi|>\varepsilon} \frac{\mathcal{F}^{-1}g(\xi) - \mathcal{F}^{-1}g(0)}{|\xi|^4}\, d\xi \\ &= \lim_{\varepsilon\to 0+} \int_{|\xi|>\varepsilon} |\xi|^{-4} \left(\frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} g(x) e^{i\xi\cdot x}\, dx - \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} g(x)\, dx \right) d\xi \\ &= \frac{1}{(2\pi)^3} \lim_{\varepsilon\to 0+} \int_{|\xi|>\varepsilon} \int_{\mathbb{R}^3} |\xi|^{-4} g(x) (e^{i\xi\cdot x} - 1)\, dx\, d\xi \\ &= \frac{1}{(2\pi)^3} \lim_{\varepsilon\to 0+} \int_{\mathbb{R}^3} g(x) \int_{|\xi|>\varepsilon} |\xi|^{-4} (e^{i\xi\cdot x} - 1)\, d\xi\, dx.\end{aligned}$$

In last equality we used Fubini's theorem. This is sound since we observe that the integrand is bounded in absolute value by $2|g(x)||\xi|^{-4}$. As we are integrating outside a neighbourhood of 0, this absolutely integrable and we can use Fubini.

We will show that

$$\int_{|\xi|>\varepsilon} |\xi|^{-4}(e^{i\xi\cdot x} - 1)\, d\xi \leq C'|x|$$

for some constant C'. This would allow us to use the dominated convergence theorem because $|x|g$ is an integrable function. For this, we will pass to the spherical coordinates. For $x \neq 0$ we have

$$\begin{aligned}
\int_{|\xi|>\varepsilon} |\xi|^{-4}(e^{i\xi\cdot x} - 1)\, d\xi &= \int_{\varepsilon}^{\infty} \int_{S^2} r^2 r^{-4}(e^{iru\cdot x} - 1)\, dS(u)\, dr \\
&= \int_{\varepsilon}^{\infty} \int_{S^2} r^{-2}(e^{iu\cdot rx} - 1)\, dS(u)\, dr \\
&= \int_{\varepsilon|x|}^{\infty} \int_{S^2} (y/|x|)^{-2}(e^{iu\cdot(y/|x|)x} - 1)\, dS(u)\, \frac{dy}{|x|} \\
&= |x| \int_{\varepsilon|x|}^{\infty} y^{-2} \int_{S^2} (e^{iyu\cdot(x/|x|)} - 1)\, dS(u)\, dy.
\end{aligned}$$

We made a change of variables $r|x| = y$. Now we observe that $(x/|x|)$ is a unit vector. The value of the inner integral is independent of x as we are integrating over the whole unit sphere (we can see this by taking a rotation).

If we split this integral again and use the symmetry, we see that the limit as $\varepsilon \to 0+$ exists. In particular, the integral is bounded by $C'|x|$ and has a limit $C|x|$ for some constant C.

Using dominated convergence theorem, we finally obtain

$$\langle E, g\rangle = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} g(x) \lim_{\varepsilon\to 0+} \int_{|\xi|>\varepsilon} |\xi|^{-4}(e^{i\xi\cdot x} - 1)\, d\xi\, dx = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} g(x) C|x|\, dx.$$

This means that the fundamental solution of Δ^2 is of form $C|x|$.