FOURIER ANALYSIS. (Fall 2016)

9. EXERCISES (Fri 2.12, 10-12 in room C322)

1. Determine the fundamental solution of Laplacian in 1-dimension, i.e. find $E \in \mathcal{S}'(\mathbf{R})$ so that $\left(\frac{d}{dx}\right)^2 E = \delta_0$.

2. Use the Poisson summation formula to prove

$$\sum_{n\in\mathbf{Z}} \frac{1}{1+n^2} = \pi \frac{1+e^{-2\pi}}{1-e^{-2\pi}}$$

3. **(i)** Suppose $A : \mathbf{R}^d \to \mathbf{R}^d$ is an invertible linear map (we denote by A also its matrix). If $f \in L^1(\mathbf{R}^d)$, define $g(x) = f(Ax)$. Show that

$$\widehat{g}(\xi) = \frac{1}{|\det(A)|} \widehat{f}((A^{-1})^T \xi),$$

where $(A^{-1})^T$ is the transpose of the inverse of A.

(ii) A function $f \in L^1(\mathbf{R}^d)$ is radial if $f(x)$ depends only on $|x|$. Use (i) to show that for a radial function, the Fourier transform is radial.

(iii) Show that the result in (ii) holds also for every radial $f \in L^2(\mathbf{R}^d)$ in a sense that the Fourier transform $\widehat{f} \in L^2(\mathbf{R}^d)$ has a radial representative.

4. Show that if E is a fundamental solution of the differential operator (with constant coefficients) $P(\partial)$, then $E + H$ is also a fundamental solution, if $H \in \mathcal{S}'(\mathbf{R}^d)$ satisfies $P(\partial)H = 0$.Verify that actually all fundamental solutions of P are obtained by this manner.

5. Recall that we proved that at the function (an example of Weierstrass functions)

$$f(x) := \sum_{n=1}^{\infty} 2^{-n/2} \cos(2^n x)$$

is not differentiable at any point. Show in any case that in the sense of distributions we have

$$f'(x) = -\sum_{n=1}^{\infty} 2^{n/2} \sin(2^n x) \qquad !$$

6. Let $A = \{(x,y) : x > 0, y > 0\} \cup \{(x,y) : x < 0, y < 0\} \subset \mathbf{R}^2$.

Show that the characteristic function χ_A is a fundamental solution for the differential operator $P_1(\partial) = \frac{1}{2}\partial_1\partial_2$.

7. (i) If $0 < \gamma < d$, show that the function

$$f_\gamma(x) = \frac{1}{|x|^\gamma}, \qquad x \in \mathbf{R}^d \setminus \{0\},$$

determines a tempered distribution, by writing it as a sum of two functions, one belonging to $L^1(\mathbf{R}^d)$ and the other to $L^p(\mathbf{R}^d)$ for a suitable $p > 1$. If $\gamma > d/2$, show that one can choose $p = 2$.

(ii) Prove that

$$\widehat{f_\gamma}(\xi) = c(d,\gamma)\frac{1}{|x|^{d-\gamma}}$$

for some constant $c(d,\gamma)$.

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8*[1] Try to figure out how might a fundamental solution of Δ^2 look in $\mathbf{R}^3$!

[1]These $*$-exercises are extras for afficinadoes, not required to get full points from exercises

Hints for some of the exercises:

T.2: [Recall the Fourier transform of $f(x) = e^{-|x|}$.]

T.7: [Hints: Consider first the case $\gamma > d/2$ and use (i) to show $\widehat{f_\gamma}$ is a function. Apply Problem 2 together with the scaling property $f_\gamma(tx) = t^{-\gamma} f_\gamma(x)$.
The case $\gamma < d/2$ follows by inverse transform, and case $\gamma = d/2$ by a limiting argument.]