FOURIER ANALYSIS. (fall 2016)

MODEL SOLUTIONS FOR SET 8

Exercise 1. Assume that the sequence of measurable functions f_n is uniformly bounded, i.e. $|f_n(x)| \leq C$ for all $x \in \mathbb{R}^d$ and $n \geq 1$, and it converges at almost every point:

$$\lim f_n(x) = g(x) \qquad \text{for almost every } \ x \in \mathbb{R}^d.$$

Show that the $f_n \to g$ in the sense of distributions.

Solution 1. Fix $\varphi \in \mathcal{S}'(\mathbb{R}^d)$. We know that

$$\int_{\mathbb{R}^d} |f_n(x)\varphi(x)|\, dx \leq \int_{\mathbb{R}^d} C|\varphi(x)|\, dx < \infty,$$

so we can use the dominated convergence theorem to see that

$$\lim_{n\to\infty} \langle f_n, \varphi \rangle = \lim_{n\to\infty} \int_{\mathbb{R}^d} f_n(x)\varphi(x)\, dx = \int_{\mathbb{R}^d} \lim_{n\to\infty} f_n(x)\varphi(x)\, dx = \int_{\mathbb{R}^d} g(x)\varphi(x)\, dx = \langle g, \varphi \rangle.$$

This shows that $f_n \to g$ in the sense of distributions.

Exercise 2. Is the function $x^2 \sin(x)$ the Fourier transform of a distribution ? If so, determine the distribution.

Solution 2. We may compute that

$$\begin{aligned}
< x^2 \sin(x), g > &= \int_{\mathbb{R}} x^2 \sin(x) g(x) dx \\
&= \int_{\mathbb{R}} \frac{1}{2i} \left(e^{ix} - e^{-ix}\right) x^2 g(x) dx \\
&= \frac{1}{2i} \left(\widehat{(x^2 g)}(-1) - \widehat{(x^2 g)}(1) \right) \\
&= \frac{i}{2} \left(\frac{d^2 \widehat{g}}{dx^2}(-1) - \frac{d^2 \widehat{g}}{dx^2}(1) \right) \\
&= \frac{i}{2} \left\langle \delta''_{-1} - \delta''_1, \widehat{g} \right\rangle . \\
&= \frac{i}{2} \left\langle \widehat{\left(\delta''_{-1} - \delta''_1\right)}, g \right\rangle
\end{aligned}$$

Thus $x^2 \sin x = \widehat{T}$, where $T = \frac{i}{2}(\delta''_{-1} - \delta''_1)$. Here of course δ is the Dirac delta distribution.

Exercise 3. **(i)** Let $f \in \mathcal{S}(\mathbb{R})$. Show that in the metric of the space $\mathcal{S}(\mathbb{R})$ it holds that $f_\varepsilon(x) \to f'(x)$ as $\varepsilon \to 0^+$, where $f_\varepsilon(x) := \varepsilon^{-1}\big(f(x+\varepsilon) - f(x)\big)$.

(ii) Use part (i) to verify that in a similar manner for any $f \in L^1(\mathbb{R})$

$$\varepsilon^{-1}\big(f(x+\varepsilon) - f(x)\big) \to \frac{d}{dx}f \quad \text{as} \quad \varepsilon \to 0,$$

where $\frac{d}{dx}f$ is the derivative of f in the sense of distributions.

Solution 3. (i) Fix N, and note that

$$p_N(f_\varepsilon - f') = \sup_{n \leq N} \sup_{x \in \mathbb{R}} (1+|x|^2)^N \left| \frac{1}{\varepsilon}(f^{(n)}(x+\varepsilon) - f^{(n)}(x)) - f^{(n+1)}(x) \right|.$$

We now estimate the expression inside. By the mean value theorem there exists $y \in [x, x+\varepsilon]$ so that

$$\frac{1}{\varepsilon}(f^{(n)}(x+\varepsilon) - f^{(n)}(x)) = f^{(n+1)}(y).$$

Similarly there exists $z \in [x, y]$ such that

$$f^{(n+1)}(y) - f^{(n+1)}(x) = f^{(n+2)}(z)(y-x).$$

Thus

$$\left| \frac{1}{\varepsilon}(f^{(n)}(x+\varepsilon) - f^{(n)}(x)) - f^{(n+1)}(x) \right| = \left| f^{(n+2)}(z) \right| |y-x| \leq \varepsilon \left| f^{(n+2)}(z) \right|.$$

Note also that since $|z - x| \leq \varepsilon$, we have for sufficiently small ε that

$$(1+|x|^2)^N \leq 2^N (1+|z|^2)^N.$$

The exact value of the constant 2^N here doesn't really matter, but the proof of this estimate can be done as follows:

$$|x| \leq |z| + \varepsilon \;\Rightarrow\; |x|^2 \leq |z|^2 + 2\varepsilon|z| + \varepsilon \;\Rightarrow\; 1+|x|^2 \leq 1 + 2\varepsilon + (1+\varepsilon)|z|^2 \leq 2(1+|z|^2).$$

By combining everything we finally get that

$$\begin{aligned} p_N(f_\varepsilon - f') &\leq \sup_{n \leq N} \sup_{z \in \mathbb{R}} 2^N (1+|z|^2)^N \varepsilon \left| f^{(n+2)}(z) \right| \\ &\leq \varepsilon 2^N p_{N+2}(f). \end{aligned}$$

This shows that $f_\varepsilon \to f'$ in the topology of $\mathcal{S}(\mathbb{R})$ as $\varepsilon \to 0$.

(ii) Let $g \in \mathcal{S}(\mathbb{R})$. We compute that

$$\begin{aligned} < f_\varepsilon, g > &= \int_{-\infty}^{\infty} \frac{1}{\varepsilon}(f(x+\varepsilon) - f(x))g(x)dx \\ &= \int_{-\infty}^{\infty} \frac{1}{\varepsilon}(g(x-\varepsilon) - g(x))f(x)dx \\ &= \int_{-\infty}^{\infty} g_{-\varepsilon} f(x)dx. \end{aligned}$$

Applying part (i) for the function $h(x) = g(-x)$ shows that the functions $g_{-\varepsilon}$ converge uniformly to $-g'$ as $\varepsilon \to 0$ so we can use dominated convergence to conclude that

$$\lim_{\varepsilon \to 0} < f_\varepsilon, g > = -\int_{-\infty}^{\infty} g'(x) f(x) dx = \; - < f, g' > = < f', g > .$$

This shows that $\lim_{\varepsilon \to 0} f_\varepsilon = f'$ in $\mathcal{S}'(\mathbb{R})$.

Exercise 4. **(i)** Show that $< \mathcal{F}^{-1} T, g > = < T, \mathcal{F}^{-1} g >$ for all $T \in \mathcal{S}'(\mathbb{R}^d)$ and $g \in \mathcal{S}(\mathbb{R}^d)$.

(ii) Verify that $\mathcal{F}^4 \lambda = (2\pi)^{2d} \lambda$ for any $\lambda \in \mathcal{S}'(\mathbb{R}^d)$.

Solution 4. **(i)** Let us show that defining the inverse Fourier transform $\mathcal{F}^{-1}$ on $\mathcal{S}'$ by

$$\left\langle \mathcal{F}^{-1} T, g \right\rangle = \left\langle T, \mathcal{F}^{-1} g \right\rangle$$

actually gives an inverse of the Fourier transform. This is easily seen since

$$\left\langle \mathcal{F}^{-1} \widehat{T}, g \right\rangle = \left\langle \widehat{T}, \mathcal{F}^{-1} g \right\rangle = \left\langle T, \mathcal{F}\mathcal{F}^{-1} g \right\rangle = \langle T, g \rangle$$

and

$$\left\langle \mathcal{F}\mathcal{F}^{-1} T, g \right\rangle = \left\langle \mathcal{F}^{-1} T, \widehat{g} \right\rangle = \left\langle T, \mathcal{F}^{-1} \widehat{g} \right\rangle = \langle T, g \rangle .$$

(ii) Recall that for any $g \in \mathcal{S}(\mathbb{R}^d)$ we have $(\mathcal{F}^2 g)(x) = (2\pi)^d g(-x)$, which implies that $(\mathcal{F}^4 g)(x) = (2\pi)^{2d} g(x)$. Now we simply compute

$$\left\langle \mathcal{F}^4 \lambda, g \right\rangle = \left\langle \lambda, \mathcal{F}^4 g \right\rangle = \left\langle \lambda, (2\pi)^{2d} g \right\rangle = \left\langle (2\pi)^{2d} \lambda, g \right\rangle .$$

Exercise 5. Let $K \in L^1$ with $\int_{\mathbb{R}^d} K(x) dx = 1$ and set $K_\varepsilon(x) := \varepsilon^{-d} K(x/\varepsilon)$ for any $\varepsilon > 0$. Prove that in the sense of distributions

$$\lim_{\varepsilon \to 0^+} K_\varepsilon = \delta_0.$$

Solution 5. Fix $\varphi \in \mathcal{S}'(\mathbb{R}^d)$ and $\nu > 0$. As φ is continuous, there is some $\delta > 0$ such that $|\varphi(x) - \varphi(0)| < \nu$ whenever $|x| < \delta$.

Now we compute

$$\begin{aligned}
|\langle K_\varepsilon, \varphi \rangle - \langle \delta_0, \varphi \rangle| &= \left| \int_{\mathbb{R}^d} K_\varepsilon(x) \varphi(x) \, dx - \varphi(0) \right| = \left| \int_{\mathbb{R}^d} K_\varepsilon(x) [\varphi(x) - \varphi(0)] \, dx \right| \\
&\le \int_{|x|<\delta} |K_\varepsilon(x)| |\varphi(x) - \varphi(0)| \, dx + \int_{|x| \ge \delta} |K_\varepsilon(x)| |\varphi(x) - \varphi(0)| \, dx \\
&\le \nu \int_{|x|<\delta} |K_\varepsilon(x)| \, dx + 2 p_0(\varphi) \int_{|x| \ge \delta} |K_\varepsilon(x)| \, dx \\
&\le \nu \int_{\mathbb{R}^d} |K(x)| \, dx + 2 p_0(\varphi) \int_{|x| \ge \delta/\varepsilon} |K(x)| \, dx
\end{aligned}$$

In last step we used the change of variables. The first term is bounded by $\nu\|K\|_{L^1}$ and the second term converges to 0 by the dominated convergence theorem. This means that

$$\limsup_{\varepsilon\to 0+} |\langle K_\varepsilon, \varphi\rangle - \langle \delta_0, \varphi\rangle| \le \nu\|K\|_{L^1}.$$

As ν was arbitrary, the claim follows.

Exercise 6. Show that $f(x) = \log|x| \in \mathcal{S}'(\mathbb{R})$ and that the distributional derivative of f is

$$\frac{d}{dx}(\log|x|) = \mathbf{p.v.}\frac{1}{x}$$

Solution 6. The function $\log|x|$ is L^1-integrable around $x = 0$ and grows slower than a polynomial as $|x| \to \infty$. This easily shows that it defines a tempered distribution on $\mathbb{R}$. We now compute that

$$\begin{aligned}
\left\langle \frac{d}{dx}\log|x|, g\right\rangle &= -\left\langle \log|x|, \frac{d}{dx}g\right\rangle \\
&= -\int_{-\infty}^{\infty} \log|x| g'(x)dx \\
&= -\int_{-\infty}^{0} \log(-x)g'(x)dx - \int_{0}^{\infty}\log(x)g'(x)dx \\
&= -\lim_{\varepsilon\to 0}\int_{-1/\varepsilon}^{-\varepsilon}\log(-x)g'(x)dx - \lim_{\varepsilon\to 0}\int_{\varepsilon}^{1/\varepsilon}\log(x)g'(x)dx \\
&= \lim_{\varepsilon\to 0}\left(-\log(\varepsilon)g(-\varepsilon) + \log(1/\varepsilon)g(-1/\varepsilon) - \log(1/\varepsilon)g(1/\varepsilon) + \log(\varepsilon)g(\varepsilon)\right) \\
&\qquad + \lim_{\varepsilon\to 0}\int_{-1/\varepsilon}^{-\varepsilon}\frac{1}{x}g(x)dx + \lim_{\varepsilon\to 0}\int_{\varepsilon}^{1/\varepsilon}\frac{1}{x}g(x)dx \\
&= 0 + \left\langle \text{p.v.}\frac{1}{x}, g\right\rangle
\end{aligned}$$

We have used integration by parts here, and dominated convergence to conclude

$$\int_{-\infty}^{0}\log(-x)g'(x)\,dx = \lim_{\varepsilon\to 0}\int_{-1/\varepsilon}^{-\varepsilon}\log(-x)g'(x)\,dx$$

and

$$\int_{0}^{\infty}\log(x)g'(x)\,dx = \lim_{\varepsilon\to 0}\int_{\varepsilon}^{1/\varepsilon}\log(x)g'(x)\,dx.$$

Additionally, since g is in $S(\mathbb{R})$ we were able to conclude that

$$\lim_{\varepsilon\to 0}\log(1/\varepsilon)g(-1/\varepsilon) = \lim_{\varepsilon\to 0}\log(1/\varepsilon)g(1/\varepsilon) = 0$$

and

$$\lim_{\varepsilon\to 0}\left(\log(\varepsilon)g(\varepsilon) - \log(\varepsilon)g(-\varepsilon)\right) = \lim_{\varepsilon\to 0}\varepsilon\log(\varepsilon)\frac{g(\varepsilon) - g(-\varepsilon)}{\varepsilon} = 0.$$

Thus $\frac{d}{dx}\log|x| = \text{p.v.}\frac{1}{x}$.

Exercise 7. Let $\psi \in C_0^\infty(\mathbb{R})$. Determine the Fourier transform of the distribution λ, where

$$\langle \lambda, g \rangle := \int_{\mathbb{R}} \psi(u) g(u,0)\, du \qquad \text{for all} \quad g \in \mathcal{S}(\mathbb{R}^2).$$

Solution 7. By the definition of the Fourier transform for distributions,

$$\begin{aligned}
\langle \widehat{\lambda}, g \rangle = \langle \lambda, \widehat{g} \rangle &= \int_{\mathbb{R}} \psi(u) \widehat{g}(u,0)\, du \\
&= \int_{\mathbb{R}} \psi(u) \int_{\mathbb{R}^2} e^{iux} g(x,y)\, dx\, dy\, du \\
&= \int_{\mathbb{R}}^2 g(x,y) \int_{\mathbb{R}} \psi(u) e^{iux}\, du\, dx\, dy \\
&= \int_{\mathbb{R}}^2 g(x,y) \widehat{\psi}(x)\, dx\, dy.
\end{aligned}$$

We have shown that $\widehat{\lambda}$ is a function with

$$\widehat{\lambda}(x,y) = \widehat{\psi}(x).$$

Exercise 8*. (i) Define $h(x) := \displaystyle\int_0^x \frac{\sin t}{t} dt$. Show that $h : [0,\infty) \to \mathbb{R}$ is a bounded function.

(ii) Determine $\lim_{x\to\infty} h(x) = \displaystyle\int_0^\infty \frac{\sin t}{t} dt$ by considering the function

$$g(t) := \frac{1}{\sin(t/2)} - \frac{2}{t}.$$

Solution 8*. (i) First of all, h is continuous and $h(0) = 1$. We see that h has extrema in points $x = n\pi$ for any positive integer n. Now, if we consider the sequence $a_n = h(n\pi) - h((n-1)\pi)$, we see that $a_n > 0$ exactly when n is odd. Additionally, $|a_n| \leq \pi/(n-1)$, so $a_n \to 0$ as $n \to \infty$.

We also see that the sequence $|a_n|$ is decreasing:

$$|a_n| - |a_{n+1}| = \int_0^\pi |\sin f| \left(\frac{1}{n\pi + x} - \frac{1}{n\pi + \pi + x} \right) dt > 0.$$

We therefore know that there exists a limit of extreme values $\lim_{n\to\infty} a_n$, so the function has a limit at infinity and is therefore bounded.

(ii) Notice that by a change of variables $t = Mx$ we obtain that

$$\int_0^{\pi M} \frac{\sin t}{t} dt = \int_0^\pi \frac{\sin(Mx)}{x} dx = \frac{1}{2} \int_{-\pi}^\pi \frac{\sin(Mx)}{x} dx. \tag{1}$$

This identity is useful because we will be able to calculate the limit of the expression on the right hand side as $M \to \infty$. First of all, we know that for each positive integer N it holds that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(x)dx = 1 \quad \Leftrightarrow \quad \int_{-\pi}^{\pi} \frac{\sin((N+1/2)x)}{\sin(x/2)}dx = 2\pi.$$

Secondly, we consider the function $g(x) = 1/\sin(x/2) - 2/x$. We show that g is continuous on the interval $[-\pi, \pi]$. On this interval, $\sin(x/2)$ is nonzero except for $x = 0$. At this point we have the Taylor series expansion

$$\sin\frac{x}{2} = \frac{x}{2} + \varepsilon(x)x^3,$$

where $\varepsilon(x)$ is bounded around $x = 0$. Thus

$$\lim_{x\to 0}\left(\frac{1}{\sin\frac{x}{2}} - \frac{2}{x}\right) = \lim_{x\to 0}\frac{x - 2\sin\frac{x}{2}}{x\sin\frac{x}{2}} = \lim_{x\to 0} -\frac{2\varepsilon(x)x^3}{x(x/2+\varepsilon(x)x^3)} = 0.$$

We now see that

$$\int_{-\pi}^{\pi}\sin((N+1/2)x)g(x)dx = \frac{1}{2i}\int_{-\pi}^{\pi}\left(e^{i(N+1/2)x} - e^{-i(N+1/2)x}\right)g(x)dx,$$

where the right hand side converges to zero as $N \to \infty$ by an application of the Riemann-Lebesgue lemma to the functions

$$g(x)e^{ix/2} \quad \text{and} \quad g(x)e^{-ix/2},$$

both continuous on the interval $[-\pi, \pi]$ and thus eligible for use of the theorem. Note that here the limit is only taken over positive integers N. It follows that

$$0 = \lim_{N\to\infty}\int_{-\pi}^{\pi}\sin((N+1/2)x)g(x)dx = \lim_{N\to\infty}\int_{-\pi}^{\pi}\sin((N+1/2)x)\left(\frac{1}{\sin(x/2)} - \frac{2}{x}\right)dx, \quad (2)$$

and hence by (2) that

$$\begin{aligned} 2\pi &= \lim_{N\to\infty}\int_{-\pi}^{\pi}\frac{\sin((N+1/2)x)}{\sin(x/2)}dx \\ &= \lim_{N\to\infty}\int_{-\pi}^{\pi}\frac{\sin((N+1/2)x)}{x/2}dx \\ &= 4\lim_{N\to\infty}\int_{0}^{(N+1/2)\pi}\frac{\sin(x)}{x}dx, \end{aligned}$$

the last equality being a consequence of (1). The limit is still only over positive integers N, but we would like to replace it by a limit over real numbers to conclude that

$$2\pi = 4\lim_{M\to\infty}\int_0^{(M+1/2)\pi}\frac{\sin(x)}{x}dx = 4\lim_{M\to\infty}\int_0^{M}\frac{\sin(x)}{x}dx = 4\int_0^{\infty}\frac{\sin x}{x}dx,$$

where the limits are taken over real numbers M. The reason why we can take the limit over real numbers instead of the positive integers is because the integral

$$\int_0^\infty \frac{\sin x}{x} dx = \lim_{M \to \infty} \int_0^M \frac{\sin(x)}{x} dx$$

is known to be convergent and because for any positive number a we have that

$$\left| \int_0^{(N+1/2)\pi + a} \frac{\sin(x)}{x} dx - \int_0^{(N+1/2)\pi} \frac{\sin(x)}{x} dx \right| \leq \frac{a}{(N+1/2)\pi},$$

which converges to zero as $N \to \infty$. Hence we can always change the right endpoint of integration to a number of the form $(N + 1/2)\pi$ for integer N without changing the limit of the integral.

We have shown that $\lim_{x \to \infty} h(x) = \frac{\pi}{2}$