**FOURIER ANALYSIS. (Fall 2016)**

**8. EXERCISES (Fri 25.11, 10-12 in room C322)**

**1.** Assume that the sequence of measurable functions $f_n$ is uniformly bounded, i.e. $|f_n(x)| \leq C$ for all $x \in \mathbf{R}^n$ and $n \geq 1$, and it converges at almost every point:

$$\lim f_n(x) = g(x) \qquad \text{for almost every } \; x \in \mathbf{R}^d.$$

Show that the $f_n \to g$ in the sense of distributions.

**2.** Is the function $x^2 \sin(x)$ the Fourier transform of a distribution ? If so, determine the distribution.

**3. (i)** Let $f \in \mathcal{S}(\mathbf{R})$. Show that in the metric of the space $\mathcal{S}(\mathbf{R})$ it holds that $f_\varepsilon(x) \to f'(x)$ as $\varepsilon \to 0^+$, where $f_\varepsilon(x) := \varepsilon^{-1}\big(f(x+\varepsilon) - f(x)\big)$.

**(ii)** Use part (i) to verify that in a similar manner for any $f \in L^1(\mathbf{R})$

$$\varepsilon^{-1}\big(f(x+\varepsilon) - f(x)\big) \to \frac{d}{dx} f \quad \text{as} \quad \varepsilon \to 0,$$

where $\frac{d}{dx} f$ is the derivative of $f$ in the sense of distributions.

**4. (i)** Show that $< \mathcal{F}^{-1}T, g > = < T, \mathcal{F}^{-1}g >$ for all $T \in \mathcal{S}'(\mathbf{R}^d)$ and $g \in \mathcal{S}(\mathbf{R}^d)$.

(ii) Verify that $\mathcal{F}^4 \lambda = (2\pi)^{2d} \lambda$ for any $\lambda \in \mathcal{S}'(\mathbf{R}^d)$.

**5.** Let $K \in L^1$ with $\int_{\mathbf{R}^d} K(x)dx = 1$ and set $K_\varepsilon(x) := \varepsilon^{-d} K(x/\varepsilon)$ for any $\varepsilon > 0$. Prove that in the sense of distributions

$$\lim_{\varepsilon \to 0^+} K_\varepsilon = \delta_0.$$

**6.** Show that $f(x) = \log|x| \in \mathcal{S}'(\mathbf{R})$ and that the distributional derivative of $f$ is

$$\frac{d}{dx}(\log|x|) = \mathbf{p.v.}\frac{1}{x}$$

**7.** Let $\psi \in C_0^\infty(\mathbf{R})$. Determine the Fourier transform of the distribution $\lambda$, where

$$\langle \lambda, g \rangle := \int_{\mathbf{R}} \psi(u) g(u,0) du \qquad \text{for all} \quad g \in \mathcal{S}(\mathbf{R}^2).$$

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**8***[1] **(i)** Define $h(x) := \int_0^x \frac{\sin t}{t} dt$. Show that $h : [0, \infty) \to \mathbf{R}$ is a bounded function.

**(ii)** Determine $\lim_{x \to \infty} h(x) = \int_0^\infty \frac{\sin t}{t} dt$ by considering the function

$$g(t) := \frac{1}{\sin(t/2)} - \frac{2}{t}.$$

[1]These $*$-exercises are extras for afficinadoes, not required to get full points from exercises

**Hints for some of the exercises:**

**T.2:** [Recall what is the Fourier transform of an exponential and apply the basic rules of calculus for Fourier transforms of distributions.]

**T.5:** [Suggestion: do it directly, or prove the same convergence for the Fourier transforms!]

**T.6:** [Compute first the derivative of $\chi_{|x|\geq\varepsilon}\log(|x|)$. Use the fact that for test functions $|g(u)-g(-u)|\leq C|u|$.]

**T.8:** [Part (i) is not difficult! In part (ii), recall from the theory of the Fourier series that the Dirichlet kernel can be expressed in terms of sine functions.]