FOURIER ANALYSIS. (fall 2016)

MODEL SOLUTIONS FOR SET 7

Exercise 1. Prove in detail that $\rho(f_n, g) \to 0$ if and only if $p_N(f_n - g) \to 0$ for every $N \geq 0$.

Solution 1. Direction $\Rightarrow$. We assume that $\rho(f_n, f) \to 0$. Suppose to the contrary that there exists N_0 such that $p_{N_0}(f_n, f) \geq \epsilon$ for infinitely many n. Then for such n

$$\rho(f_n, f) = \sum_{N=0}^{\infty} 2^{-N} \frac{p_N(f_n - f)}{1 + p_N(f_n - f)} \geq 2^{-N_0} \frac{p_{N_0}(f_n - f)}{1 + p_{N_0}(f_n - f)} \geq 2^{-N_0} \frac{\epsilon}{1 + \epsilon} > 0,$$

a contradiction.

Direction $\Leftarrow$. Assume that $p_N(f_n - f) \to 0$ for all N. Let $\epsilon > 0$ and N_0 be a large number to be chosen later. We estimate that

$$\begin{aligned}
\rho(f_n, f) &= \sum_{N=0}^{\infty} 2^{-N} \frac{p_N(f_n - f)}{1 + p_N(f_n - f)} \\
&= \sum_{N=0}^{N_0} 2^{-N} \frac{p_N(f_n - f)}{1 + p_N(f_n - f)} + \sum_{N=N_0}^{\infty} 2^{-N} \frac{p_N(f_n - f)}{1 + p_N(f_n - f)} \\
&\leq \sum_{N=0}^{N_0} 2^{-N} \frac{p_N(f_n - f)}{1 + p_N(f_n - f)} + \sum_{N=N_0}^{\infty} 2^{-N} \\
&\leq \sum_{N=0}^{N_0} 2^{-N} \frac{p_N(f_n - f)}{1 + p_N(f_n - f)} + 2^{-N_0+1}.
\end{aligned}$$

If we choose N_0 large enough, the second term will be less than $\epsilon/2$. The first term can then be estimated since it contains only finitely many terms, and for each term we can use our assumption to choose $n \geq n_0$ large enough so that $p_N(f_n - f)$ is as small as we wish. Thus we can also bound the first term by $\epsilon/2$ if we want to, which proves the claim.

Exercise 2. Prove that a linear map $\lambda : \mathcal{S}(\mathbb{R}^d) \to \mathbb{C}$ is continuous if and only if there is an index $N \geq 0$ and constant $C < \infty$ such that

$$|\lambda(g)| \leq C p_N(g) \qquad \text{for all} \quad g \in \mathcal{S}(\mathbb{R}^d).$$

Solution 2. By translation invariance of λ and the metric $\rho(f, g) = \rho(f - g, 0)$, the fact that T is continuous is equivalent with the fact that it is continuous at zero. With this in mind, we proceed.

Direction $\Leftarrow$. Let f_n be a sequence converging to zero in the metric ρ. By Exercise 1 we know that $p_N(f_n) \to 0$ as $N \to \infty$. Since $|\lambda(f_n)| \leq C p_N(f_n)$, we also know that $\lambda(f_n) \to 0$ so λ is continuous at zero.

Direction $\Rightarrow$. We make a proof by contradiction. Suppose that for every N and constant C there is a function $f_{N,C}$ such that

$$|\lambda(f_N)| \geq Cp_N(f_{N,C}).$$

We choose $C = N$ to get a sequence f_N of functions for which

$$|\lambda(f_N)| \geq Np_N(f_N).$$

By linearity of T, we can scale this to assume that $p_N(f_N) = 1/N$. We prove that then $f_N \to 0$ in the metric ρ. By Exercise 1 it is enough to prove that $p_M(f_N) \to 0$ for every M. But this follows from the fact that if $N \geq M$, then by the fact that the p_M are increasing in M we get

$$p_M(f_N) \leq p_N(f_N) = 1/N.$$

However,

$$|\lambda(f_N)| \geq 1,$$

so λ cannot be continuous at zero.

Exercise 3. Assume that $f \in C^\infty(\mathbb{R}^d)$ satisfies for any multi-index α: there exists $M = M_\alpha$ and $C = C_\alpha$ so that

$$|\partial^\alpha f(x)| \leq C(1+|x|^2)^M \quad \text{for all } \; x \in \mathbb{R}^d.$$

Show that $fg \in \mathcal{S}(\mathbb{R}^d)$ for all $g \in \mathcal{S}(\mathbb{R}^d)$ and that the map $g \mapsto fg$ is a continuous linear map from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}(\mathbb{R}^d)$.

Solution 3. The function fg is smooth as a product of smooth functions, and linearity is trivial. We need to estimate the norms $p_N(fg)$. To prove that $fg \in \mathcal{S}(\mathbb{R}^d)$, we need to show that the norms are finite, and for continuity we need to estimate them by the norms of g. We use the general Leibniz formula from Exercise 8 in the previous set:

$$\begin{aligned}
p_N(fg) &= \sup_{|\alpha|\leq N} \sup_{x\in\mathbb{R}^d} (1+|x|^2)^N |\partial^\alpha (fg)(x)| \\
&= \sup_{|\alpha|\leq N} \sup_{x\in\mathbb{R}^d} (1+|x|^2)^N \left| \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} \partial^\beta f(x) \partial^{\alpha-\beta} g(x) \right| \\
&\leq \sup_{|\alpha|\leq N} \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} \sup_{x\in\mathbb{R}^d} (1+|x|^2)^N \left| \partial^\beta \phi(x) \partial^{\alpha-\beta} g(x) \right| \\
&\leq \sup_{|\alpha|\leq N} \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} \sup_{x\in\mathbb{R}^d} C(1+|x|^2)^{N+M} \left| \partial^{\alpha-\beta} g(x) \right| \\
&\leq \sup_{|\alpha|\leq N} \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} Cp_{N+M}(g) \\
&= C2^N p_{N+M}(g).
\end{aligned}$$

Here the constants C and M are defined by

$$C = \max\{C_\alpha : |\alpha| \le N\} \quad \text{and} \quad M = \max\{M_\alpha : |\alpha| \le N\}.$$

Now the map $g \mapsto fg$ is continuous by Theorem 12.2. in the lecture notes.

Exercise 4. Show that the metric space $(\mathcal{S}(\mathbb{R}^d), \rho)$ (i.e. the Schwartz space of test functions equipped with the metric ρ) is complete.

Solution 4. Suppose that we have a Cauchy sequence (f_n) in the metric ρ. Then (f_n) will also be Cauchy in each of the norms p_N, which follows from the fact that

$$\frac{p_N(f_n - f_m)}{1 + p_N(f_n - f_m)} \le 2^N \sum_{N=0}^{\infty} 2^{-N} \frac{p_N(f_n - f_m)}{1 + p_N(f_n - f_m)} = 2^N \rho(f_n, f_m),$$

so if $\rho(f_n, f_m)$ is small then $p_N(f_n - f_m)$ must be small as well. Choosing $N = 0$, we find that (f_n) is Cauchy in the sup-norm and thus has a limit f in the sup-norm. For each multi-index α, the sequence $(\partial_\alpha f_n)$ is also Cauchy in the sup-norm and thus converges to some function g_α. By basic results about uniformly converging sequences of functions we know that $\partial^\alpha f = g_\alpha$ (see the course Analysis II). We must still prove that f_n converges to f in the metric ρ. Let $\epsilon > 0$ be arbitrary. Then there is n_0 such that $p_N(f_m - f_n) \le \epsilon$ when $m, n \ge n_0$. Thus

$$(1 + |x|^2)^N |\partial^\alpha f_n(x) - f_m(x)| \le \epsilon$$

By uniform convergence, we let $n \to \infty$ to get that

$$(1 + |x|^2)^N |\partial^\alpha f(x) - f_m(x)| \le \epsilon.$$

This shows that $p_N(f - f_m) \to 0$ as $m \to \infty$ for every N, enough to show the convergence in the metric ρ by (ii). Obviously also f is in the Schwartz class by $p_N(f) \le p_N(f - f_m) + p_N(f_m)$ for every N.

Exercise 5. **(i)** Let $a = (a_1, a_2) \in \mathbb{R}^2$ and $r > 0$. Show that $T \in \mathcal{S}'(\mathbb{R}^2)$, where

$$\langle T, g \rangle := \int_0^{2\pi} g(a + r(\cos(t), \sin(t))dt$$

when $g \in \mathcal{S}(\mathbb{R}^2)$.

(ii) Verify that $T \in \mathcal{S}'(\mathbb{R})$, where

$$\langle T, \phi \rangle := \sum_{k \in \mathbb{Z}} \phi(k^2).$$

Solution 5. **(i)** It is clear that T is linear. As $|g(x)| \le p_0(g)$, we can estimate

$$
\begin{aligned}
\langle T, g\rangle| &= \left|\int_0^{2\pi} g(a + r(\cos(t), \sin(t))\, dt\right| \le \int_0^{2\pi} |g(a + r(\cos(t), \sin(t))|\, dt \\
&\le \int_0^{2\pi} p_0(g), dt = 2\pi p_0(g).
\end{aligned}
$$

This proves by Exercise 2 that $T \in \mathcal{S}'(\mathbb{R})$.

(ii) The linearity is again clear. We know that $|\phi(x)| \le p_1(\phi)/(1 + |x|^2)$. We may now estimate

$$
\langle T, \phi\rangle| = \left|\sum_{k\in\mathbb{Z}} \phi(k^2)\right| \le \sum_{k\in\mathbb{Z}} |\phi(k^2)| \le \sum_{k\in\mathbb{Z}} \frac{p_1(\phi)}{1 + k^4} = Cp_1(\phi).
$$

Here $C = \sum_{k\in\mathbb{Z}}(1 + k^4)^{-1}$ is the value of the sum. Exercise 2 implies that that $T \in \mathcal{S}'(\mathbb{R})$.

Exercise 6. Let $f = \chi_{[0,1]}$ be the characteristic function of an interval. What is the distribution derivative of f?

Solution 6. The distribution associated with f is T_f with

$$
\langle T_f, \phi\rangle := \int_{\mathbb{R}} f(x)\phi(x)\, dx.
$$

Using the definition of the distribution derivative gives

$$
\langle T_f', \phi\rangle = -\langle T_f, \phi'\rangle = -\int_{\mathbb{R}} f(x)\phi'(x)\, dx = -\int_0^1 \phi'(x)\, dx = \phi(0) - \phi(1).
$$

We may write this as $f' = \delta_0 - \delta_1$ in the sense of distributions.

Exercise 7*. Show that functions that grow too fast do not necessarily define distributions. More specifically, show that $e^{|x|}$ does not define an element in $\mathcal{S}'(\mathbb{R})$ in the following sense: the map $\lambda : C_0^\infty(\mathbb{R}) \to \mathbb{C}$, where

$$
\langle \lambda, g\rangle := \int_{\mathbb{R}} e^{|x|} g(x) dx
$$

does not have a continuous extension to the space $\mathcal{S}(\mathbb{R})$.

Solution 7*. We will show that the condition given in Exercise 2 is not satisfied. Let $h \in C_c^\infty(\mathbb{R})$ be a non-trivial positive smooth function supported on the interval $[1, 2]$, and let N be arbitrary non-negative integer.

For any positive integer K we consider the function h_K defined as $h_K(x) = h(x-K)$. The function h_K is supported on the interval $[K+1, K+2]$, and we see that for any multi-index α with $|\alpha| \leq N$ we have

$$\sup_{x\in\mathbb{R}}(1+|x|^2)^N|\partial^\alpha h_K(x)| \leq \sup_{x\in\mathbb{R}}(1+(K+2)^2)^N|\partial^\alpha h(x)| \leq (1+(K+2)^2)^N p_N(h).$$

In particular, $p_N(h_K) \leq (1+(K+2)^2)^N p_N(h)$

On the other hand, we have

$$|\langle\lambda, h_K\rangle| = \int_{\mathbb{R}} e^{|x|}h_K(x)dx = e^K\int_{\mathbb{R}} e^{|x|}h(x)dx = e^K\langle\lambda, h\rangle.$$

If we had $\langle\lambda, g\rangle \leq Cp_N(g)$ for any $g \in C_c^\infty(\mathbb{R})$, then we would have for any positive integer K

$$e^K\langle\lambda, h\rangle = \langle\lambda, h_K\rangle \leq Cp_N(h_K) \leq (1+(K+2)^2)^N p_N(h),$$

or equivalently

$$(1+(K+2)^2)^{-N}e^K \leq p_N(h)/\langle\lambda, h\rangle.$$

But $p_N(h)/\langle\lambda, h\rangle$ is a constant while $(1+(K+2)^2)^{-N}e^K$ grows to infinity as $K\to\infty$. This means that the inequality $\langle\lambda, g\rangle \leq Cp_N(g)$ cannot hold for all $g \in C_c^\infty(\mathbb{R})$. As N was arbitrary, it follows that λ cannot have a continuous extension to $\mathcal{S}(\mathbb{R})$.