FOURIER ANALYSIS. (Fall 2016)

7. EXERCISES (Fri 18.11, 10-12 in room C322)

1. Prove in detail that $\rho(f_n, g) \to 0$ if and only if $p_N(f_n - g) \to 0$ for every $N \geq 0$.

2. Prove that a linear map $\lambda : \mathcal{S}(\mathbf{R}^d) \to \mathbf{C}$ is continuous if and only if there is an index $N \geq 0$ and constant $C < \infty$ such that

$$|\lambda(g)| \leq C p_N(g) \qquad \text{for all} \quad g \in \mathcal{S}(\mathbf{R}^d).$$

3. Assume that $f \in C^\infty(\mathbf{R}^d)$ satisfies for any multi-index α: there exists $M = M_\alpha$ and $C = C_\alpha$ so that

$$|\partial^\alpha f(x)| \leq C(1 + |x|^2)^M \quad \text{for all} \;\; x \in \mathbf{R}^d.$$

Show that $fg \in \mathcal{S}(\mathbf{R}^d)$ for all $g \in \mathcal{S}(\mathbf{R}^d)$ and that the map $g \mapsto fg$ is a continuous linear map from $\mathcal{S}(\mathbf{R}^d)$ to $\mathcal{S}(\mathbf{R}^d)$.

4. Show that the metric space $(\mathcal{S}(\mathbf{R}^d), \rho)$ (i.e. the Schwartz space of test functions equipped with the metric ρ) is complete.

5. **(i)** Let $a = (a_1, a_2) \in \mathbf{R}^2$ and $r > 0$. Show that $T \in \mathcal{S}'(\mathbf{R}^2)$, where

$$\langle T, g \rangle := \int_0^{2\pi} g(a + r(\cos(t), \sin(t))dt$$

when $g \in \mathcal{S}(\mathbf{R}^d)$.

(ii) Verify that $T \in \mathcal{S}'(\mathbf{R})$, where

$$\langle T, \phi \rangle := \sum_{k \in \mathbf{Z}} \phi(k^2).$$

6. Let $f = \chi_{[0,1]}$ be the characteristic function of an interval. What is the distribution derivative of f ?

7*[1] Show that functions that grow too fast do not necessarily define distributions. More specifically, show that $e^{|x|}$ does not define an element in $\mathcal{S}'(\mathbf{R})$ in the following sense: the map $\lambda : C_0^\infty(\mathbf{R}) \to \mathbf{C}$, where

$$\langle \lambda, g \rangle := \int_{\mathbf{R}} e^{|x|} g(x) dx$$

does not have a continuous extension to the space $\mathcal{S}(\mathbf{R})$.

[1] These $*$-exercises are extras for afficinadoes, not required to get full points from exercises

Hints for some of the exercises:

T.4: [Hint: Apply on each derivative the fact that if functions and their derivatives converge uniformly, then the limit function is differentiable and the derivative is the limit of derivatives.]