FOURIER ANALYSIS. (fall 2016)

MODEL SOLUTIONS FOR SET 6

Exercise 1. Let $\alpha \in \mathbb{N}_0^d$ be a multi-index. Prove with all details that if $f \in \mathcal{S}(\mathbb{R}^d)$, then

(i) $x^\alpha f(x) \in \mathcal{S}(\mathbb{R}^d)$ and $\partial^\alpha f(x) \in \mathcal{S}(\mathbb{R}^d)$,

(ii) $\widehat{f} \in C^\infty(\mathbb{R}^d)$.

(iii) $(\partial^\alpha f)\hat{}(\xi) = (i\,\xi)^\alpha \widehat{f}(\xi)$ (note that one defines $i^\alpha := i^{|\alpha|}$).

(iv) Apply part (iii) by choosing suitable multi-indices α to verify that $\widehat{f}$ decays any polynomial rate, i.e. for any $N \geq 1$ there is a constant C so that $|\widehat{f}(\xi)| \leq C(1+|\xi|^2)^{-N}$.

Solution 1. For ease of notation, let's say that a function f satisfies the $(*)$**-condition** if

$$\sup_{x\in\mathbb{R}^d}(1+|x|)^N|f(x)| < \infty \quad \text{for all } N \geq 0.$$

Thus a function is in $\mathcal{S}(\mathbb{R}^d)$ if it and all of its derivatives satisfy the $(*)$-condition.

(i) If α is a multi-index, we recall that

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_d^{\alpha_d}.$$

By induction it will be enough to show that

$$x_j f \in \mathcal{S}(\mathbb{R}^d)$$

for every j. Let us calculate the partial derivatives of $x_j f$. For $k \neq j$ we have

$$\partial_k(x_j f) = x_j \partial_k f$$

and

$$\partial_j(x_j f) = f + x_j \partial_j f.$$

Using this, we see that if α is a multi-index with $\alpha_j = 0$, then

$$\partial^\alpha x_j f = x_j \partial^\alpha f$$

. If $\alpha_j \neq 0$, then we let $\alpha' = \alpha - e_j$ be the multi-index with jth coordinate one less than α and all other coordinates equal. Then we see that

$$\partial^\alpha x_j f = x_j \partial^\alpha f + \alpha_j \partial^{\alpha'} f.$$

We now prove that if a function g satisfies the $(*)$-condition, then $x_j g$ satisfies it as well. After proving this we see that all the partial derivatives of $x_j f$ also satisfy $(*)$-condition, so $x_j f \in \mathcal{S}(\mathbb{R}^d)$. To see that $x_j g$ satisfies $(*)$-condition, we estimate

$$\sup_{x\in\mathbb{R}^d}(1+|x|)^N|x_jg(x)| \le \sup_{x\in\mathbb{R}^d}(1+|x|)^{N+1}|g(x)| < \infty.$$

It is also easy to see that $\partial^\alpha f$ is in $\mathcal{S}(\mathbb{R}^d)$, since $\partial^\beta\partial^\alpha f = \partial^{\beta+\alpha}f$, which satisfies $(*)$-condition for every β.

(ii) We apply Theorem 9.4 and induction to show that $\partial^\alpha\widehat{f}(\xi) = ((-ix)^\alpha f(x))\widehat{\ }(\xi)$. We assume that the formula holds for some multi-index α. Let $e_j = (0,\dots,0,1,0,\dots,0)$, where jth index is 1. According to part (i) $g(x) = (-ix_j)(-ix)^\alpha f(x) \in \mathcal{S}(\mathbb{R}^d) \subset L^1(\mathbb{R}^d)$ and therefore

$$\begin{aligned}\partial^{\alpha+e_j}\widehat{f}(\xi) &= \frac{\partial}{\partial\xi_j}(\partial^\alpha\widehat{f}(\xi)) = \frac{\partial}{\partial\xi_j}((-ix)^\alpha f(x))\widehat{\ }(\xi)\\ &= (-ix_j(-ix)^\alpha f(x))\widehat{\ }(\xi) = ((-ix)^{\alpha+e_j}f(x))\widehat{\ }(\xi).\end{aligned}$$

The formula therefore holds for $\alpha+e_j$, and by induction, for any multi-index. In particular, $\widehat{f}\in C^\infty(\mathbb{R}^d)$.

(iii) By induction it is enough to prove that

$$(\partial_j f)\widehat{\ }(\xi) = (i\xi_j)\widehat{f}(\xi)$$

for all j. To do this we use integration by parts to obtain that

$$\int_{\mathbb{R}^d}(\partial_j f)(x)e^{-i\xi\cdot x}dx = -\int_{\mathbb{R}^d}f(x)\partial_j e^{-i\xi\cdot x}dx = i\xi_j\int_{\mathbb{R}^d}f(x)e^{-i\xi\cdot x}dx,$$

which is what we wanted.

(iv) Let $N \ge 1$ be fixed. For any fixed multi-index α we know by part (i) that $\partial^\alpha f \in \mathcal{S}(\mathbb{R}^d)$. Since the Fourier transform maps any $L^1(\mathbb{R}^d)$ function into $L^\infty(\mathbb{R}^d)$, we know by part (iii) that $(i\,\xi)^\alpha\,\widehat{f}(\xi)$ is in $L^\infty(\mathbb{R}^d)$ for every α. Hence we get the bounds

$$|\xi^\alpha||\widehat{f}(\xi)| \le C_\alpha.$$

As $(1+|\xi|^2)^N$ is a polynomial, we can express it as a finite sum of terms of form ξ^α. Hence there exists a constant C_N such that

$$(1+|\xi|^2)^N|\widehat{f}(\xi)| \le C_N.$$

Dividing by $(1+|\xi|^2)^N$ gives $|\widehat{f}(\xi)| \le C(1+|\xi|^2)^{-N}$ as wanted.

Exercise 2. Apply the previous exercise and verify carefully that

$$\text{if}\quad f\in\mathcal{S}(\mathbb{R}^d),\quad\text{then}\quad \widehat{f}\in\mathcal{S}(\mathbb{R}^d).$$

Solution 2. We proved in the previous exercise that $\widehat{f} \in C^{\infty}(\mathbb{R}^d)$. It remains to show that all the derivatives satisfy the $(*)$-condition.

The previous exercise already implies that for any Schwartz function g the function $\widehat{g}$ already satisfies the $(*)$-condition. We also showed that for any multi-index α

$$\partial^{\alpha}\widehat{f}(\xi) = ((-ix)^{\alpha} f(x))\widehat{\ }(\xi)$$

and $(-ix)^{\alpha} f(x)$ is a Schwartz function. We see that any derivative of $\widehat{f}$ is Fourier transform of a Schwartz function and therefore satisfies the $(*)$-condition.

Exercise 3. Which of the following functions belong to $\mathcal{S}(\mathbb{R}^d)$?

(i) $f(x) = (1 + |x|^2)^{-1}$. **(ii)** $f(x) = e^{-|x|^2}$.

(iii) $f(x) = e^{-|x|^2} \cos(e^{|x|^2})$.

Solution 3. **(i)** This function does not belong in $\mathcal{S}(\mathbb{R}^d)$. We see that

$$\sup_{x \in \mathbb{R}^d} (1 + |x|^2)^2 |f(x)| = \sup_{x \in \mathbb{R}^d} 1 + |x|^2 = \infty.$$

(ii) This function belongs in $\mathcal{S}(\mathbb{R}^d)$.

The function is smooth as it is a composition of smooth functions. Note that $f(x)$ itself satisfies $(*)$-condition, since the exponential function grows faster than any polynomial. Now

$$\partial_j f(x) = -2x_j e^{-|x|^2}.$$

By induction we see that $\partial^{\alpha} f(x)$ is some polynomial times $f(x)$ for every multi-index α. But these types of functions also satisfy $(*)$-condition, since we proved that if f satisfies $(*)$-condition then $x_j f$ also satisfies $(*)$-condition and we can use induction to prove this for any polynomial in place of x_j. Thus all the partial derivatives satisfy $(*)$-condition and $f \in \mathcal{S}(\mathbb{R}^d)$.

(iii) This function does not belong in $\mathcal{S}(\mathbb{R}^d)$. Considering its partial derivative shows that

$$\begin{aligned}
\sup_{x \in \mathbb{R}^d} (1 + |x|^2) |\partial_1 f(x)| &\geq \sup_{x \in \mathbb{R}^d} \left| -2x_1 e^{-|x|^2} \cos(e^{|}x|^2) - e^{-|x|^2} \sin(e^{|}x|^2) e^{|x|^2} 2x_1 \right| \\
&= \sup_{x \in \mathbb{R}^d} \left| -2x_1 e^{-|x|^2} \cos(e^{|}x|^2) - 2x_1 \sin(e^{|}x|^2) \right| = \infty,
\end{aligned}$$

because the term $2x_1 \sin(e^{|}x|^2)$ is not bounded.

Exercise 4. Compute the integral $\displaystyle\int_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^2 dx$ by first computing the Fourier transform of the characteristic function $\chi_{[-1,1]}$.

Solution 4. Recall from Exercise 1 in the previous set that

$$\widehat{\chi}_{[-1,1]} = \frac{2\sin(\xi)}{\xi}.$$

We also know that for any $f \in L^2(\mathbb{R})$

$$2\pi \int_{\mathbb{R}} |f(x)|^2 \, dx = \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 \, d\xi.$$

Now we use these facts to see that

$$4\pi = 2\pi \int_{-1}^{1} 1^2 dx = \int_{-\infty}^{\infty} \left(\frac{2\sin\xi}{\xi} \right)^2 d\xi$$

and we can solve that

$$\int_{-\infty}^{\infty} \left(\frac{\sin\xi}{\xi} \right)^2 d\xi = \pi.$$

Exercise 5. Assume that $f \in \mathcal{S}(\mathbb{R}^d)$. **(i)** Compute the Fourier transform of the Laplacian $\Delta f := \big(\sum_{j=1}^d \big(\frac{\partial}{\partial x_j}\big)^2 \big) f$ in terms of $\widehat{f}$.

(ii) Show that $\dfrac{f(x)}{1+|x|^2} \in \mathcal{S}(\mathbb{R}^d)$.

Solution 5. **(i)** Using (iii) from Exercise 1, we find that

$$\widehat{\Delta f}(\xi) = \sum_{j=1}^{d} \widehat{\partial_{x_j}^2 f}(\xi) = \sum_{j=1}^{d} (i\xi_j)^2 \widehat{f}(\xi) = -|\xi|^2 \widehat{f}(\xi).$$

(ii) Let us say that a function $R(x)$ is a **good rational function** if

$$R(x) = \frac{P(x)}{Q(x)},$$

where $P(x)$ and $Q(x)$ are polynomials and $Q(x)$ does not take the value zero. Especially $(1+|x|^2)^{-1}$ is a good rational function. The claim of the exercise now follows from these two results:

Claim 1. If $R(x)$ is a good rational function and f satisfies $(*)$-condition from Exercise 1, then $R(x)f(x)$ satisfies $(*)$-condition.

Proof. If $R(x)$ is a good rational function, then $|R(x)| \leq C(1+|x|)^M$ for some constants $C, M \geq 0$. Thus

$$\sup_{x\in\mathbb{R}^d} (1+|x|)^N |R(x)f(x)| \leq \sup_{x\in\mathbb{R}^d} C(1+|x|)^{N+M} |f(x)| < \infty$$

for all N. This proves the claim.

Claim 2. If a function is of the form $R(x)f(x)$ with R a good rational function and $f \in \mathcal{S}$, then all of its first-order derivatives are also sums of functions of the same form.

Proof. We simply compute that

$$\partial_{x_j} R(x)f(x) = \frac{(\partial_{x_j} P(x))Q(x) - P(x)\partial_{x_j} Q(x)}{Q(x)^2} f(x) + R(x)\partial_{x_j} f(x),$$

which is of the desired form.

The claim now follows by induction. By Claim 1, the function $(1+|x|^2)^{-1}f(x)$ satisfies $(*)$-condition. By Claim 2 and 1, so do its first order derivatives. Continuing this argument we find that all the derivatives satisfy $(*)$-condition, so $(1+|x|^2)^{-1}f(x) \in \mathcal{S}$.

Exercise 6. Use Fourier transform to find a solution formula for the partial differential equation

$$\Delta f - f = g$$

for given $g \in \mathcal{S}(\mathbb{R}^d)$ and show that also the solution f lies in $\mathcal{S}(\mathbb{R}^d)$.

Solution 6. If $f \in \mathcal{S}$ satisfies the equation

$$\Delta f - f = g, \qquad \Delta = \left(\frac{\partial}{\partial x_1}\right)^2 + \cdots + \left(\frac{\partial}{\partial x_d}\right)^2,$$

we can take the Fourier transform of both sides to find that

$$-(|\xi|^2 + 1)\widehat{f}(\xi) = \widehat{g}(\xi).$$

Recall from Exercise 2 that we also know that $\widehat{f}(\xi), \widehat{g}(\xi) \in \mathcal{S}$. We can now solve the Fourier transform of f:

$$\widehat{f}(\xi) = -(1+|\xi|^2)^{-1}\widehat{g}(\xi).$$

By previous exercise, we know that $-(1+|\xi|^2)\widehat{g}(\xi) \in \mathcal{S}$. Thus it is possible to take the inverse Fourier transform $\mathcal{F}^{-1}$ to find the solution f:

$$f(x) = \mathcal{F}^{-1}\left[-(1+|\xi|^2)^{-1}\widehat{g}(\xi)\right](x).$$

This solves the original equation so we are done.

Exercise 7. **(i)** Specialize in the previous exercise to dimension $d = 1$ and show that the solution is given by the convolution

$$f(x) = -\frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-y|} g(y)dy.$$

(ii) Given $\varepsilon > 0$, show that one may pick $g \in \mathcal{S}(\mathbb{R}^d)$ so that the solution f satisfies $\|f\|_{L^2(\mathbb{R})} < \varepsilon \|g\|_{L^2(\mathbb{R})}$.

Solution 7. **(i)** We define function h as $h(x) = e^{-|x|}$. We saw in Exercise 3 of the previous set that $\widehat{h}(\xi) = \frac{2}{1+\xi^2}$. Therefore

$$\widehat{f}(\xi) = -\frac{1}{2}\widehat{g}(\xi)\widehat{h}(\xi).$$

As the Fourier transform of the convolution is the product of the Fourier transforms, we have

$$f(x) = -\frac{1}{2}(g * h)(x) = -\frac{1}{2}\int_{-\infty}^{\infty} g(y)e^{-|x-y|}\,dy.$$

(ii) Let M be a constant. Let h be a C_c^∞ function with $h(x) = 0$ for $|x| \leq M$ that is not identically 0. As compactly supported smooth functions are Schwartz functions, we know that there exists a Schwartz function g with $\widehat{g} = h$.

Now we can estimate

$$\begin{aligned}\|f\|_{L^2} &= (2\pi)^{-1/2}\|\widehat{f}\|_{L^2} = (2\pi)^{-1/2}\left(\int (1+|\xi|^2)^{-2}|\widehat{g}(\xi)|^2 d\xi\right)^{1/2} \\ &\leq (2\pi)^{-1/2}\left(\int (1+M^2)^{-2}|\widehat{g}(\xi)|^2 d\xi\right)^{1/2} = (1+M^2)^{-1}(2\pi)^{-1/2}\|\widehat{g}\|_{L^2} \\ &= (1+M^2)^{-1}\|g\|_{L^2}\end{aligned}$$

As $\|g\| \neq 0$ and M was arbitrary, the claim follows.

Exercise 8*. Prove Leibniz general rule for differentiation of products: if $\alpha \in \mathbb{N}_0^d$ is an arbitrary multi-index and $f, g \in C^\infty(\mathbb{R}^d)$, then

$$\partial^\alpha(fg)(x) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^\beta f(x)\, \partial^{\alpha-\beta} g(x),$$

where $\displaystyle\binom{\alpha}{\beta} := \prod_{j=1}^{d} \binom{\alpha_j}{\beta_j}$

Solution 8*. We prove the claim by induction. The case $\alpha = 0$ is trivial. Assume that the claim holds for any $|\alpha| \leq n$, $n \geq 0$. Then if $|\alpha| = n+1$, we may write $\alpha = \beta + e_i$ for some $i \in \{1, \ldots, d\}$ and $\beta \in \mathbb{N}_0^d$ ($|\beta| = n$). Here e_i is a multi-index with $(e_i)_j = 1$ if $j = i$ and

$(e_i)_j = 0$ otherwise. Now using induction hypothesis we get

$$
\begin{aligned}
\partial^\alpha (fg)(x) &= \partial^{e_i}\partial^\beta (fg)(x) = \partial^{e_i} \sum_{\gamma\le\beta} \binom{\beta}{\gamma} \partial^\gamma f(x) \partial^{\beta-\gamma} g(x) \\
&= \sum_{\gamma\le\beta} \binom{\beta}{\gamma} \partial^{e_i}\partial^\gamma f(x)\partial^{\beta-\gamma} g(x) + \sum_{\gamma\le\beta} \binom{\beta}{\gamma} \partial^\gamma f(x)\partial^{e_i}\partial^{\beta-\gamma} g(x) \\
&= \sum_{\gamma\le\beta} \binom{\beta}{\gamma} \partial^{\gamma+e_i} f(x)\partial^{\beta+e_i-(\gamma+e_i)} g(x) + \sum_{\gamma\le\beta} \binom{\beta}{\gamma} \partial^\gamma f(x)\partial^{\beta+e_i-\gamma} g(x) \\
&= \sum_{\gamma\le\beta+e_i,\gamma_i\ge1} \binom{\beta}{\gamma-e_i} \partial^\gamma f(x)\partial^{\alpha-\gamma} g(x) + \sum_{\gamma\le\beta,\gamma_i\ge1} \binom{\beta}{\gamma} \partial^\gamma f(x)\partial^{\alpha-\gamma} g(x) \\
&\quad + \sum_{\gamma\le\beta,\gamma_i=0} \binom{\beta}{\gamma} \partial^\gamma f(x)\partial^{\alpha-\gamma} g(x) \\
&= \sum_{\gamma\le\beta,\gamma_i\ge1} \binom{\beta}{\gamma-e_i} \partial^\gamma f(x)\partial^{\alpha-\gamma} g(x) + (\partial^\alpha f(x))g(x) \\
&\quad + \sum_{\gamma\le\beta,\gamma_i\ge1} \binom{\beta}{\gamma} \partial^\gamma f(x)\partial^{\alpha-\gamma} g(x) + \sum_{\gamma\le\beta,\gamma_i=0} \binom{\beta}{\gamma} \partial^\gamma f(x)\partial^{\alpha-\gamma} g(x) \\
&= \sum_{\gamma\le\beta,\gamma_i\ge1} \left(\binom{\beta}{\gamma-e_i} + \binom{\beta}{\gamma} \right) \partial^\gamma f(x)\partial^{\alpha-\gamma} g(x) + (\partial^\alpha f(x))g(x) + \\
&\quad \sum_{\gamma\le\beta,\gamma_i=0} \binom{\beta}{\gamma} \partial^\gamma f(x)\partial^{\alpha-\gamma} g(x) \\
&= \sum_{\gamma\le\beta,\gamma_i\ge1} \left(\binom{\beta_i}{\gamma_i-1} + \binom{\beta_i}{\gamma_i} \right) \prod_{j\ne i} \binom{\beta_j}{\gamma_j} \partial^\gamma f(x)\partial^{\alpha-\gamma} g(x) \\
&\quad + (\partial^\alpha f(x))g(x) + \sum_{\gamma\le\beta,\gamma_i=0} \binom{\beta}{\gamma} \partial^\gamma f(x)\partial^{\alpha-\gamma} g(x) \\
&= \sum_{\gamma\le\beta,\gamma_i\ge1} \binom{\alpha_i}{\gamma_i} \prod_{j\ne i} \binom{\alpha_j}{\gamma_j} \partial^\gamma f(x)\partial^{\alpha-\gamma} g(x) \\
&\quad + (\partial^\alpha f(x))g(x) + \sum_{\gamma\le\beta,\gamma_i=0} \binom{\alpha}{\gamma} \partial^\gamma f(x)\partial^{\alpha-\gamma} g(x) \\
&= \sum_{\gamma\le\beta} \binom{\alpha}{\gamma} \partial^\gamma f(x)\partial^{\alpha-\gamma} g(x) + (\partial^\alpha f(x))g(x) \\
&= \sum_{\gamma\le\alpha} \binom{\alpha}{\gamma} \partial^\gamma f(x)\partial^{\alpha-\gamma} g(x).
\end{aligned}
$$