FOURIER ANALYSIS. (Fall 2016)

6. EXERCISES (Fri 11.11, 10-12 in room C322)

1. Let $\alpha \in \mathbf{N}_0^d$ be a multi-index. Prove with all details that if $f \in \mathcal{S}(\mathbf{R}^d)$, then

(i) $x^\alpha f(x) \in \mathcal{S}(\mathbf{R}^d)$ and $\partial^\alpha f(x) \in \mathcal{S}(\mathbf{R}^d)$,

(ii) $\widehat{f} \in C^\infty(\mathbf{R}^d)$.

(iii) $(\partial^\alpha f)\widehat{\ }(\xi) = (i\,\xi)^\alpha \widehat{f}(\xi)$ (note that one defines $i^\alpha := i^{|\alpha|}$).

(iv) Apply part (iii) by choosing suitable multi-indices α to verify that $\widehat{f}$ decays any polynomial rate, i.e. for any $N \geq 1$ there is a constant C so that $|\widehat{f}(\xi)| \leq C(1+|\xi|^2)^{-N}$.

2. Apply the previous exercise and verify carefully that

$$\text{if} \quad f \in \mathcal{S}(\mathbf{R}^d), \quad \text{then} \quad \widehat{f} \in \mathcal{S}(\mathbf{R}^d).$$

3. Which of the following functions belong to $\mathcal{S}(\mathbf{R}^d)$?

(i) $f(x) = (1+|x|^2)^{-1}$. **(ii)** $f(x) = e^{-|x|^2}$.

(iii) $f(x) = e^{-|x|^2}\cos(e^{|x|^2})$.

4. Compute the integral $\int_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^2 dx$ by first computing the Fourier transform of the characteristic function $\chi_{[-1,1]}$.

5. Assume that $f \in \mathcal{S}(\mathbf{R}^d)$. **(i)** Compute the Fourier transform of the Laplacian $\Delta f := \left(\sum_{j=1}^d \left(\frac{\partial}{\partial x_j}\right)^2\right) f$ in terms of $\widehat{f}$.

(ii) Show that $\frac{f(x)}{1+|x|^2} \in \mathcal{S}(\mathbf{R}^d)$.

6. Use Fourier transform to find a solution formula for the partial differential equation

$$\Delta f - f = g$$

for given $g \in \mathcal{S}(\mathbf{R}^d)$ and show that also the solution f lies in $\mathcal{S}(\mathbf{R}^d)$.

7. **(i)** Specialize in the previous exercise to dimension $d = 1$ and show that the solution is given by the convolution

$$f(x) = -\frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-y|} g(y)\,dy.$$

(ii) Given $\varepsilon > 0$, show that one may pick $g \in \mathcal{S}(\mathbf{R}^d)$ so that the solution f satisfies $\|f\|_{L^2(\mathbf{R})} < \varepsilon \|g\|_{L^2(\mathbf{R})}$.

8^{*1} Prove Leibniz general rule for differentiation of products: if $\alpha \in \mathbf{N}_0^d$ is an arbitrary multi-index and $f, g \in C^\infty(\mathbf{R}^d)$, then

$$\partial^\alpha (fg)(x) \;=\; \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \partial^\beta f(x)\, \partial^{\alpha-\beta} g(x),$$

where $\displaystyle \binom{\alpha}{\beta} \;:=\; \prod_{j=1}^{d} \binom{\alpha_j}{\beta_j}$

[1]These $*$-exercises are extras for afficinadoes, not required to get full points from exercises

Hints for some of the exercises:

T.1: [(i) Suggestion: show that $x_j\phi(x) \in \mathcal{S}(\mathbf{R}^d)$ if $\phi \in \mathcal{S}(\mathbf{R}^d)$ and apply induction. Or alternatively, use Leibniz rule for the derivative of a product.]

T.4: [Hint: Recall the L^2-theory for the Fourier transform!]

T.6: [Hint:use exercise 5.]

T.7: [Hint: in part (i) recall Exercise 3/set 5.]