FOURIER ANALYSIS. (fall 2016)

MODEL SOLUTIONS FOR SET 5

Exercise 1. Compute the Fourier transform of the characteristic function $\chi_{[-a,a]}$ (here $a > 0$).

Solution 1. By definition,

$$\widehat{\chi}_{[-a,a]}(\xi) = \int_{-a}^{a} e^{-i\xi x}\, dx = \frac{e^{-i\xi a} - e^{i\xi a}}{-i\xi} = \frac{2\sin(a\xi)}{\xi}.$$

Exercise 2. (i) Compute the convolution $\chi_{[-a,a]} * \chi_{[-a,a]}$.

(ii) Compute the Fourier transform (in one dimension) of the function $g(x) = \max(0, 1 - |x|)$.

Solution 2. (i) A straightforward computation shows that

$$\chi_{[-a,a]} * \chi_{[-a,a]}(x) = \int_{-a}^{a} \chi_{[-a,a]}(x-y)\, dy = \max(0, 2a - |x|).$$

(ii) By (i) we see that $g(x) = \chi_{[-1/2,1/2]} * \chi_{[-1/2,1/2]}(x)$. Using Exercise 1, we get

$$\widehat{g}(\xi) = (\widehat{\chi}_{[-1/2,1/2]}(\xi))^2 = \frac{4\sin^2(\xi/2)}{\xi^2}$$

Exercise 3. Compute the Fourier transform $f : \mathbb{R} \to \mathbb{R}$, where $f(x) := e^{-k|x|}$ (here $k > 0$) : show that

$$\widehat{f}(\xi) = \frac{2k}{k^2 + \xi^2}$$

Solution 3. This is a pretty straightforward computation

$$\begin{aligned}
\widehat{f}(\xi) &= \int_{-\infty}^{\infty} e^{-i\xi x} e^{-k|x|} dx \\
&= \int_{0}^{\infty} e^{-(i\xi + k)x} dx + \int_{-\infty}^{0} e^{-(i\xi - k)x} dx \\
&= \left[\frac{1}{-(i\xi + k)} e^{-(i\xi + k)x}\right]_0^{\infty} + \left[\frac{1}{-(i\xi - k)} e^{-(i\xi - k)x}\right]_{-\infty}^{0} \\
&= \frac{1}{i\xi + k} - \frac{1}{i\xi - k} \\
&= \frac{2k}{k^2 + \xi^2}.
\end{aligned}$$

Here we have used the facts that

$$\lim_{x \to \infty} e^{-(i\xi + k)x} = 0 \quad \text{and} \quad \lim_{x \to -\infty} e^{-(i\xi - k)x} = 0,$$

following from $|e^{-(i\xi + k)x}| = e^{-kx}$ and $|e^{-(i\xi - k)x}| = e^{kx}$.

Exercise 4. **(i)** If $f \in L^1(\mathbb{R}^d)$ and $g(x) = \overline{f(-x)}$, show that $\widehat{g}(\xi) \equiv \overline{\widehat{f}(\xi)}$.

(ii) If $f \in L^1(\mathbb{R}^d)$ and $g(x) = \frac{1}{t^d} f\left(\frac{x}{t}\right)$, $t > 0$, show that $\widehat{g}(\xi) \equiv \widehat{f}(t\,\xi)$.

Solution 4. **(i)** We compute simply that

$$\begin{aligned}\widehat{g}(\xi) &= \int_{\mathbb{R}^d} e^{-i\xi\cdot x}\overline{f(-x)}dx \\ &= \int_{\mathbb{R}^d} e^{i\xi\cdot x}\overline{f(x)}dx \\ &= \int_{\mathbb{R}^d} \overline{e^{-i\xi\cdot x} f(x)}dx \\ &= \overline{\int_{\mathbb{R}^d} e^{-i\xi\cdot x} f(x)dx} \\ &= \overline{\widehat{f}(\xi)}.\end{aligned}$$

(ii) Using the change of variables $y = x/t = (x_1/t, x_2/t, \ldots, x_n/t)$ whose Jacobian determinant is equal to $1/t^d$, we find that

$$\begin{aligned}\widehat{g}(\xi) &= \int_{\mathbb{R}^d} e^{-i\xi\cdot x} f\left(\frac{x}{t}\right)\frac{1}{t^d}dx \\ &= \int_{\mathbb{R}^d} e^{-i\xi\cdot ty} f(y)dy \\ &= \int_{\mathbb{R}^d} e^{-it\xi\cdot y} f(y)dy \\ &= \widehat{f}(t\xi).\end{aligned}$$

Exercise 5. Suppose that the function $f : \mathbb{R}^d \to \mathbb{C}$ has the form

$$f(x) = f_1(x_1)f_2(x_2)\cdots f_d(x_d), \qquad \forall\, x = (x_1, \ldots, x_d) \in \mathbb{R}^d,$$

where $f_1, \ldots, f_d \in L^1(\mathbb{R}^1)$. Show that then $f \in L^1(\mathbb{R}^d)$ and we have

$$\widehat{f}(\xi) = \widehat{f_1}(\xi_1)\widehat{f_2}(\xi_2)\cdots\widehat{f_d}(\xi_d) \qquad \forall\, \xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d.$$

Solution 5. Applying Fubini's theorem for non-negative functions, we see that $\int_{\mathbb{R}^d} |f(x)|\,dx = \int_{-\infty}^{\infty} |f_1(x_1)|\,dx_1 \int_{-\infty}^{\infty} |f_2(x_2)|\,dx_2 \cdots \int_{-\infty}^{\infty} |f_d(x_d)|\,dx_d < \infty$. As $f \in L^1(\mathbb{R}^d)$, we can now apply Fubini's theorem and compute the Fourier transform. If $d > 1$, we can define a function $g : \mathbb{R}^{d-1} \to \mathbb{C}$ by setting $g(x) = f_1(x_1)f_2(x_2)\cdots f_{d-1}(x_{d-1})$ and denote $\widetilde{\xi} =$

$(\xi_1, \ldots, \xi_{d-1})$. We may then compute

$$\begin{aligned}
\widehat{f}(\xi) &= \int_{\mathbb{R}^d} f(x) e^{-i\xi \cdot x}\, dx \\
&= \int_{-\infty}^{\infty} \int_{\mathbb{R}^{d-1}} g(y) f_d(x_d) e^{-i(\widetilde{\xi} y + \xi_d x_d)}\, dy\, dx_d \\
&= \int_{-\infty}^{\infty} f_d(x_d) e^{-i\xi_d x_d} \int_{\mathbb{R}^{d-1}} g(y) e^{-i\widetilde{\xi} \cdot y}\, dy\, dx_d \\
&= \int_{-\infty}^{\infty} f_d(x_d) e^{-i\xi_d x_d} \widehat{g}(\widetilde{\xi})\, dx_d \\
&= \widehat{f_d}(\xi_d) \widehat{g}(\widetilde{\xi}).
\end{aligned}$$

Now g is a function defined on $\mathbb{R}^{d-1}$ so we can use induction to deduce

$$\widehat{f}(\xi) = \widehat{f_1}(\xi_1) \widehat{f_2}(\xi_2) \cdots \widehat{f_d}(\xi_d)$$

Exercise 6. Assume that $H \in L^1(\mathbb{R}^d)$ fulfils $H \geq 0$, and $\int_{\mathbb{R}^d} H(x)\, dx = 1$, together with

$$|H(x)| \leq \frac{C}{(1 + |x|)^{d+1}}.$$

For $\varepsilon > 0$ let us denote $H_\varepsilon(x) := \varepsilon^{-d} H(x/\varepsilon)$. If $f \in L^1(\mathbb{R}^d)$ is continuous at 0, prove that

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} f(x) H_\varepsilon(x)\, dx \;=\; f(0).$$

Solution 6. Let $\eta > 0$ be arbitrary. By continuity of f we can find $\delta > 0$ such that $|f(x) - f(0)| < \eta$ when $|x| < \delta$.

Now as $\int_{\mathbb{R}^d} H_\varepsilon(x)\, dx = \int_{\mathbb{R}^d} H(x)\, dx = 1$, we have

$$\left| \int_{\mathbb{R}^d} f(x) H_\varepsilon(x)\, dx - f(0) \right| = \left| \int_{\mathbb{R}^d} (f(x) - f(0)) H_\varepsilon(x)\, dx \right|.$$

Split the integral in three parts:

$$\int_{B(0,\delta)} (f(x) - f(0)) H_\varepsilon(x)\, dx + \int_{\mathbb{R}^d \setminus B(0,\delta)} f(x) H_\varepsilon(x)\, dx - \int_{\mathbb{R}^d \setminus B(0,\delta)} f(0) H_\varepsilon(x)\, dx.$$

Then in the first integral $|x| < \delta$, so $|f(x) - f(0)| < \eta$. Now we have

$$\left| \int_{B(0,\delta)} (f(x) - f(0)) H_\varepsilon(x)\, dx \right| \leq \int_{B(0,\delta)} \eta H_\varepsilon(x)\, dx \leq \eta.$$

We can estimate the second integral as

$$\begin{aligned}\left|\int_{\mathbb{R}^d\setminus B(0,\delta)} f(x)H_\varepsilon(x)\,dx\right| &= \left|\int_{\mathbb{R}^d\setminus B(0,\delta)} f(x)\frac{H(x/\varepsilon)}{\varepsilon^d}\,dx\right| \\ &\leq \int_{\mathbb{R}^d\setminus B(0,\delta)} |f(x)|\frac{C}{\varepsilon^d(1+|x/\varepsilon|)^{d+1}}\,dx \\ &\leq \int_{\mathbb{R}^d\setminus B(0,\delta)} |f(x)|\frac{C}{(1+\delta/\varepsilon)^{d+1}}\frac{1}{\varepsilon^d}\,dx \\ &\leq \int_{\mathbb{R}^d\setminus B(0,\delta)} |f(x)|C\frac{\varepsilon}{(\varepsilon+\delta)^{d+1}}\,dx \\ &\leq \|f\|_{L^1}C\frac{\varepsilon}{\delta^{d+1}} \to 0.\end{aligned}$$

For the third integral, we can apply dominated convergence theorem to compute

$$\begin{aligned}\left|\int_{\mathbb{R}^d\setminus B(0,\delta)} f(0)H_\varepsilon(x)\,dx\right| &= \left|\int_{\mathbb{R}^d\setminus B(0,\delta)} f(0)\frac{H(x/\varepsilon)}{\varepsilon^d}\,dx\right| \\ &= \left|\int_{\mathbb{R}^d\setminus B(0,\delta/\varepsilon)} f(0)H(x)dx\right| \\ &\leq \int_{\mathbb{R}^d\setminus B(0,\delta/\varepsilon)} |f(0)|H(x)\,dx \to 0.\end{aligned}$$

We have shown that

$$\limsup_{\varepsilon\to 0}\left|\int_{\mathbb{R}^d} f(x)H_\varepsilon(x)\,dx - f(0)\right| \leq \eta.$$

As $\eta > 0$ was arbitrary, we have shown that

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^d} f(x)H_\varepsilon(x)\,dx = f(0).$$

Exercise 7. Let $a > 0$. Check that the function $H(x) := c_a e^{-a|x|^2}$ with a suitable constant c_a satisfies the conditions of the previous exercise. What is the value of c_a?

Solution 7. As the real exponential function is always positive, we have that $H(x) \geq 0$ if $c_a \geq 0$.

For the property $\int_{\mathbb{R}^d} H(x)\,dx = 1$, we compute the integral of $e^{-a|x|^2}$ and then choose c_a appropriately. The integral is slightly modified Gaussian integral, so we use Fubini's theorem:

$$\int_{\mathbb{R}^d} e^{-a|x|^2}\,dx = \left(\int_{-\infty}^{\infty} e^{-ax^2}\,dx\right)^d = \sqrt{\frac{\pi}{a}}^d.$$

This means that the integral of H is correct if we choose $c_a = (\frac{a}{\pi})^{d/2}$.

It remains to prove for all $x \in \mathbb{R}^d$ the estimate

$$|H(x)| \leq \frac{C}{(1+|x|)^{d+1}}.$$

For this, observe that it is sufficient to prove for any non-negative real x that

$$c_a e^{-ax^2} \leq \frac{C}{(1+x)^{d+1}}.$$

Consider the function $g : [0, \infty) \to \mathbb{R}$, $g(x) = (1+x)^{d+1} e^{-ax^2}$. Using l'Hospital's rule, we see that $\lim_{x\to\infty} g(x) = 0$. This means that there is a constant M such that $g(x) \leq 1$ for $x > M$. As g is continuos, it has a finite maximum in the interval $[0, M]$, and therefore g is a bounded function.

We have shown that H satisfies the conditions of the previous exercise with constant $c_a = (\frac{a}{\pi})^{d/2}$.

Exercise 8*. Prove the formula $\displaystyle\sum_{n=0}^{\infty} \frac{\cos(nx)}{n!} = e^{\cos x} \cos(\sin(x))$.

Solution 8*. Using the Taylor expansion of the exponential function we have

$$\exp(e^{ix}) = \sum_{n=0}^{\infty} \frac{e^{inx}}{n!} = \sum_{n=0}^{\infty} \frac{\cos(nx)}{n!} + i \sum_{n=0}^{\infty} \frac{\sin(nx)}{n!}.$$

Similarly

$$\exp(e^{-ix}) = \sum_{n=0}^{\infty} \frac{e^{-inx}}{n!} = \sum_{n=0}^{\infty} \frac{\cos(nx)}{n!} - i \sum_{n=0}^{\infty} \frac{\sin(nx)}{n!}.$$

So

$$\frac{\exp(e^{ix}) + \exp(e^{-ix})}{2} = \sum_{n=0}^{\infty} \frac{\cos(nx)}{n!}.$$

And

$$\frac{\exp(e^{ix}) + \exp(e^{-ix})}{2} = \frac{e^{\cos x} e^{i \sin x} + e^{\cos x} e^{-i \sin x}}{2} = e^{\cos x} \cos(\sin(x)).$$