FOURIER ANALYSIS. (Fall 2016)

4. EXERCISES (Fri 21.10, 10-12 in room C322)

1. Compute the Fourier transform of the characteristic function $\chi_{[-a,a]}$ (here $a > 0$).

2. (i) Compute the convolution $\chi_{[-a,a]} * \chi_{[-a,a]}$.

(ii) Compute the Fourier transform (in one dimension) of the function $g(x) = \max(0, 1 - |x|)$.

3. Compute the Fourier transform $f : \mathbf{R} \to \mathbf{R}$, where $f(x) := e^{-k|x|}$ (here $k > 0$) : show that

$$\widehat{f}(\xi) = \frac{2k}{k^2 + \xi^2}$$

4. (i) If $f \in L^1(\mathbf{R}^d)$ and $g(x) = \overline{f(-x)}$, show that $\widehat{g}(\xi) \equiv \overline{\widehat{f}(\xi)}$.

(ii) If $f \in L^1(\mathbf{R}^d)$ and $g(x) = \frac{1}{t^d} f\left(\frac{x}{t}\right)$, $t > 0$, show that $\widehat{g}(\xi) \equiv \widehat{f}(t\,\xi)$.

5. Suppose that the function $f : \mathbf{R}^d \to \mathbf{C}$ has the form

$$f(x) = f_1(x_1) f_2(x_2) \cdots f_d(x_d), \qquad \forall\, x = (x_1, \ldots, x_d) \in \mathbf{R}^d,$$

where $f_1, \ldots, f_d \in L^1(\mathbf{R}^1)$. Show that then $f \in L^1(\mathbf{R}^d)$ and we have

$$\widehat{f}(\xi) = \widehat{f_1}(\xi_1) \widehat{f_2}(\xi_2) \cdots \widehat{f_d}(\xi_d) \qquad \forall\, \xi = (\xi_1, \ldots, \xi_d) \in \mathbf{R}^d.$$

6. Assume that $H \in L^1(\mathbf{R}^d)$ fulfils $H \geq 0$, and $\int_{\mathbf{R}^d} H(x)\,dx = 1$, together with

$$|H(x)| \leq \frac{C}{(1 + |x|)^{d+1}}.$$

For $\varepsilon > 0$ let us denote $H_\varepsilon(x) := \varepsilon^{-d} H(x/\varepsilon)$. If $f \in L^1(\mathbf{R}^d)$ is continuous at 0, prove that

$$\lim_{\varepsilon \to 0} \int_{\mathbf{R}^d} f(x) H_\varepsilon(x)\,dx \;=\; f(0).$$

7. Let $a > 0$. Check that the function $H(x) := c_a e^{-a|x|^2}$ with a suitable constant c_a satisfies the conditions of the previous exercise. What is the value of c_a?

8*[1]

Prove the formula $\displaystyle\sum_{n=0}^{\infty} \frac{\cos(nx)}{n!} = e^{\cos x} \cos(\sin(x)).$

[1]These $*$-exercises are extras for afficinadoes, not required to get full points from exercises

Hints for some of the exercises:

T.1: [Hint: use directly the definition of the Fourier transform.]

T.2: [Hint: for part (ii) recall what is the Fourier transform of convolution and apply part (i) with suitable a.]