FOURIER ANALYSIS. (fall 2016)

MODEL SOLUTIONS FOR SET 4

Exercise 1. (i) Show that if $f_n \to f$ and $g_n \to g$ in $L^2(-\pi, \pi)$ (i.e. converging in the L^2-norm), then

$$(f_n, g_n)_{L^2} \to (f, g)_{L^2} \qquad \text{as } n \to \infty.$$

(ii) Prove the Pythagorean theorem in $L^2(-\pi, \pi)$, that is, show that

$$\|f+g\|_{L^2}^2 = \|f\|_{L^2}^2 + \|g\|_{L^2}^2 \qquad \text{if} \quad f \perp g \quad \text{and } f, g \in L^2(-\pi, \pi).$$

Solution 1. (i) Note that if f_n is a converging sequence in L^2, then it is also a bounded sequence in L^2. This is because

$$||f_n||_2 \leq ||f||_2 + ||f - f_n||_2,$$

and the right hand side remains bounded as $n \to \infty$. Hence $||f_n||_2 \leq M$ for some constant M. We now write that

$$(f_n, g_n)_{L^2} - (f, g)_{L^2} = (f_n, g_n - g)_{L^2} + (f_n - f, g)_{L^2}.$$

By the Cauchy-Schwartz inequality, $(a, b)_{L^2} \leq ||a||_2||b||_2$ for all $a, b \in L^2$, so we get that

$$|(f_n, g_n)_{L^2} - (f, g)_{L^2}| \leq M||g_n - g||_2 + ||f_n - f||_2||g||_2.$$

The right hand side converges to zero as $n \to \infty$, so the left hand side must converge to zero too. This proves the result.

(ii) If $f \perp g$, then $(f, g)_{L^2} = 0$. The result is now proven with the simple computation

$$\begin{aligned} ||f+g||_2^2 &= (f+g, f+g)_{L^2} \\ &= (f, f+g)_{L^2} + (g, f+g)_{L^2} \\ &= (f, f)_{L^2} + (f, g)_{L^2} + \overline{(f, g)_{L^2}} + (g, g)_{L^2} \\ &= (f, f)_{L^2} + (g, g)_{L^2} \\ &= ||f||_2^2 + ||g||_2^2. \end{aligned}$$

Exercise 2. Suppose $f \in C^1_\#(-\pi, \pi)$. Show that the Fourier series of f converges absolutely, i.e. we have $\sum |\widehat{f}(n)| < \infty$.

Solution 2. Suppose f is in $C^1_\#(-\pi, \pi)$. Then the Fourier coefficients of f' are well-defined and for all $n \in \mathbb{Z}$ we have the formula

$$\widehat{f'}(n) = in\widehat{f}(n).$$

We now apply the Cauchy-Schwartz inequality to the two sequences

$$(1/n)_{n=1}^{\infty} \quad \text{and} \quad (|\widehat{f'}(n)|)_{n=1}^{\infty}.$$

The first one is obviously in ℓ^2, and the second one is too since by Plancherel's formula

$$\sum_{n=-\infty}^{\infty} |\widehat{f'}(n)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f'(x)|^2 dx < \infty.$$

Thus we have that

$$\begin{aligned}
\sum_{n=1}^{\infty} |\widehat{f}(n)| &= \sum_{n=1}^{\infty} \frac{1}{n} |\widehat{f'}(n)| \\
&\leq \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \right)^{1/2} \left(\sum_{n=1}^{\infty} |\widehat{f'}(n)|^2 \right)^{1/2} \\
&< \infty.
\end{aligned}$$

Similarly we see that

$$\sum_{n=-\infty}^{-1} |\widehat{f}(n)| < \infty.$$

Hence

$$\sum_{n=-\infty}^{\infty} |\widehat{f}(n)| = \sum_{n=-\infty}^{-1} |\widehat{f}(n)| + |\widehat{f}(0)| + \sum_{n=1}^{\infty} |\widehat{f}(n)| < \infty.$$

Thus the Fourier series of f converges absolutely.

Exercise 3. (i) Show that for every 2π-periodic function $f \in L^1[-\pi, \pi]$ we have

$$\widehat{f}(n) = \frac{1}{4\pi} \int_0^{2\pi} e^{-inx} \big(f(x) - f(x + \pi/n) \big) dx.$$

(ii) If $f \in C_{\#}(-\pi, \pi)$ is Hölder-continuous with exponent $\alpha \in (0, 1]$, show that

$$|\widehat{f}(n)| \leq C |n|^{-\alpha}, \quad \text{for } |n| \geq 1.$$

Solution 3. (i) Making a change of variables $x = t + \pi/n$ we find that

$$\begin{aligned}
\widehat{f}(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inx} f(x)\, dx \\
&= \frac{1}{2\pi} \int_{-\pi-\pi/n}^{\pi-\pi/n} e^{-in(t+\pi/n)} f(t + \pi/n)\, dt \\
&= -\frac{1}{2\pi} \int_{-\pi-\pi/n}^{\pi-\pi/n} e^{-int} f(t + \pi/n)\, dt \\
&= -\frac{1}{2\pi} \int_0^{2\pi} e^{-int} f(t + \pi/n)\, dt
\end{aligned}$$

We were able to replace the interval $[-\pi - \pi/n, \pi - \pi/n]$ with $[0, 2\pi]$ in the last step by the 2π-periodicity of the function $e^{-int} f(t + \pi/n)$. Now we find the required formula by the calculation

$$\begin{aligned}\widehat{f}(n) &= \frac{1}{2}\left(\widehat{f}(n) + \widehat{f}(n)\right) \\ &= \frac{1}{2}\left[\frac{1}{2\pi}\int_0^{2\pi} e^{-inx} f(x)\,dx + \left(-\frac{1}{2\pi}\int_0^{2\pi} e^{-int} f(t+\pi/n)\,dt\right)\right] \\ &= \frac{1}{4\pi}\int_0^{2\pi} e^{-inx}\big(f(x) - f(x+\pi/n)\big)\,dx.\end{aligned}$$

(ii) Suppose f is Hölder-continuous with exponent α. Then

$$|f(x) - f(x+\pi/n)| \le \left|\frac{\pi}{n}\right|^{\alpha} = \pi^{\alpha}|n|^{-\alpha},$$

and hence by part (i)

$$|\widehat{f}(n)| \le \frac{1}{4\pi}\int_0^{2\pi} |f(x) - f(x+\pi/n)|\,dx \le \frac{1}{4\pi}\int_0^{2\pi} \pi^{\alpha}|n|^{-\alpha} dx = \frac{1}{2}\pi^{\alpha}|n|^{-\alpha}.$$

This proves what we wanted.

Exercise 4. Let $f \in L^2(-\pi, \pi)$. Find the trigonometric polynomial $p(x) := \sum_{n=-N}^{N} c_n e^{inx}$ which is closest to f in L^2-norm, i.e. find the coefficients c_n that minimise the quantity

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|f(x) - \sum_{n=-N}^{N} c_n e^{inx}\right|^2 dx$$

Solution 4. Using Plancherel's formula we see that if we denote $c_n = 0$ for any $|n| > N$, we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|f(x) - \sum_{n=-N}^{N} c_n e^{inx}\right|^2 dx = \sum_{n=-\infty}^{\infty} |\widehat{f}(n) - c_n|^2 \ge \sum_{|n|>N} |\widehat{f}(n)|^2.$$

The equality holds if $|\widehat{f}(n) - c_n| = 0$ for any $|n| \le N$, in other words when $c_n = \widehat{f}(n)$. So the closest trigonometric polynomial in L^2-norm is the partial sum of the Fourier series.

Exercise 5. Assume that $f \in C^2_{\#}$ and $\int_{-\pi}^{\pi} f(x)dx = 0$. Prove the inequality

$$\int_{-\pi}^{\pi} |f(x)|^2 dx \;\le\; \int_{-\pi}^{\pi} |f''(x)|^2 dx.$$

When do you have equality here?

Solution 5. We apply Plancherel's formula to see that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2\,dx = \sum_{n=-\infty}^{\infty}|\widehat{f}(n)|^2.$$

As $\widehat{f'}(n) = in\widehat{f}(n)$, we also have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|f''(x)|^2\,dx = \sum_{n=-\infty}^{\infty}|(in)^2\widehat{f}(n)|^2 = \sum_{n=-\infty}^{\infty}n^4|\widehat{f}(n)|^2.$$

We can then use the estimate $n^4 \geq 1$ for any $n \neq 0$ to get

$$\int_{-\pi}^{\pi}|f''(x)|^2\,dx = 2\pi\sum_{n=-\infty}^{\infty}n^4|\widehat{f}(n)|^2 \geq 2\pi\sum_{n\neq 0}|\widehat{f}(n)|^2$$

Because $\int_{-\pi}^{\pi} f(x)dx = 0$, we know that $\widehat{f}(0) = 0$. This means that

$$\int_{-\pi}^{\pi}|f''(x)|^2\,dx \geq 2\pi\sum_{n\neq 0}|\widehat{f}(n)|^2 = 2\pi\sum_{n=-\infty}^{\infty}|\widehat{f}(n)|^2 = \int_{-\pi}^{\pi}|f(x)|^2\,dx.$$

For equality to hold, we must have $n^4|\widehat{f}(n)|^2 = |\widehat{f}(n)|^2$ for all n. This in particular means that $\widehat{f}(n) = 0$ whenever $|n| \neq 1$. So the equality can only hold when $f(x) = \widehat{f}(1)e^{ix} + \widehat{f}(-1)e^{-ix}$. We see that for any such function f the equality does indeed hold.

Exercise 6. Compute the Fourier series of $f(x) = x^2$, $x \in (-\pi, \pi)$ and compute the L^2-norm of f in two ways: first by direct computation and then using the Fourier-coefficients. Use this to compute the $\sum_{n=1}^{\infty} n^{-4}$.

Solution 6. A direct computation shows that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}x^4\,dx = \frac{\pi^4}{5}.$$

Next, we compute the Fourier coefficients $\widehat{f}(n)$. For $n = 0$ we have

$$\widehat{f}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi}x^2\,dx = \frac{\pi^2}{3}.$$

For $n \neq 0$ we use integration by parts:

$$\begin{aligned}
\widehat{f}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 e^{-inx}\,dx \\
&= \frac{1}{2\pi}\left[\frac{1}{-in}(\pi^2 e^{-in\pi} - (-\pi)^2 e^{in\pi}) - \int_{-\pi}^{\pi}\frac{2x}{-in}e^{-inx}\,dx\right] \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{2x}{in}e^{-inx}\,dx \\
&= \frac{1}{2\pi}\left[\frac{1}{n^2}(2\pi e^{-in\pi} - 2(-\pi)e^{in\pi}) - \int_{-\pi}^{\pi}\frac{2}{n^2}e^{-inx}\,dx\right] \\
&= \frac{2(-1)^n}{n^2}.
\end{aligned}$$

Using Plancherel's formula, we know that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2\,dx = \sum_{n=-\infty}^{\infty}|\widehat{f}(n)|^2.$$

This means that we have

$$\frac{\pi^4}{5} = \left(\frac{\pi^2}{3}\right)^2 + \sum_{n\neq 0}\left|\frac{2(-1)^n}{n^2}\right|^2 = \frac{\pi^4}{9} + 2\sum_{n=1}^{\infty}\frac{4}{n^4}.$$

We can now solve that

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\frac{\pi^4}{5}-\frac{\pi^4}{9}}{8} = \frac{\pi^4}{90}$$

Exercise 7*. Can you compute $\sum_{n=1}^{\infty} n^{-6}$ with the help of Fourier-series?

Solution 7*. We consider the function $f(x) = x^3 - \pi^2 x$ and compute its Fourier coefficients $\widehat{f}(n)$. For $n = 0$ we have

$$\widehat{f}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi}(x^3 - \pi^2 x)\,dx = 0.$$

For $n \neq 0$ we use integration by parts

$$\begin{aligned}
\widehat{f}(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (x^3 - \pi^2 x) e^{-inx}\, dx \\
&= \frac{1}{2\pi} \left[0 - 0 - \int_{-\pi}^{\pi} \frac{3x^2 - \pi^2}{-in} e^{-inx}\, dx \right] \\
&= \frac{1}{2\pi} \left[\int_{-\pi}^{\pi} \frac{3x^2}{in} e^{-inx}\, dx - \int_{-\pi}^{\pi} \frac{\pi^2}{in} e^{-inx}\, dx \right] \\
&= \frac{1}{2\pi} \left[\frac{3\pi^2}{n^2}(-1)^n - \frac{3(-\pi)^2}{n^2}(-1)^n - \int_{-\pi}^{\pi} \frac{6x}{n^2} e^{-inx}\, dx - 0 \right] \\
&= \frac{1}{2\pi} \left[\frac{6\pi}{in^3}(-1)^n - \frac{6(-\pi)}{in^3}(-1)^n - \int_{-\pi}^{\pi} \frac{6}{in^3} e^{-inx}\, dx \right] \\
&= \frac{6(-1)^n}{in^3}.
\end{aligned}$$

Using Plancehrel's formula, we now know that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2\, dx = \sum_{n=-\infty}^{\infty} |\widehat{f}(n)|^2 = \sum_{n=1}^{\infty} \frac{72}{n^6}.$$

We compute

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2\, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} (x^6 - 2\pi^2 x^4 + \pi^4 x^2)\, dx = \frac{8\pi^6}{105}.$$

We can now solve

$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}.$$

Remark. The chosen extra term $\pi^2 x$ simplified the calculations but is not necessary. We could have instead taken the function $x \to x^3$, using the fact that we already know the two sums $\sum_n n^{-2}$ and $\sum_n n^{-4}$.