**FOURIER ANALYSIS. (Fall 2016)**

**4. EXERCISES (Fri 14.10, 10-12 in room C322)**

**1. (i)** Show that if $f_n \to f$ and $g_n \to g$ in $L^2(-\pi,\pi)$ (i.e. converging in the $L^2$-norm), then

$$(f_n, g_n)_{L^2} \to (f,g)_{L^2} \qquad \text{as } n \to \infty.$$

**(ii)** Prove the Pythagorean theorem in $L^2(-\pi,\pi)$, that is, show that

$$\|f+g\|_{L^2}^2 = \|f\|_{L^2}^2 + \|g\|_{L^2}^2 \qquad \text{if} \quad f \perp g \quad \text{and} \;\; f,g \in L^2(-\pi,\pi).$$

**2.** Suppose $f \in C^1_\#(-\pi,\pi)$. Show that the Fourier series of $f$ converges absolutely, i.e. we have $\sum |\widehat{f}(n)| < \infty$.

**3. (i)** Show that for every $2\pi$-periodic function $f \in L^1[-\pi,\pi]$ we have

$$\widehat{f}(n) = \frac{1}{4\pi}\int_0^{2\pi} e^{-inx}\big(f(x) - f(x+\pi/n)\big)dx.$$

**(ii)** If $f \in C_\#(-\pi,\pi)$ is Hölder-continuous with exponent $\alpha \in (0,1]$, show that

$$|\widehat{f}(n)| \le C|n|^{-\alpha}, \quad \text{for } |n| \ge 1.$$

**4.** Let $f \in L^2(-\pi,\pi)$. Find the trigonometric polynomial $p(x) := \sum_{n=-N}^{N} c_n e^{inx}$ which is closest to $f$ in $L^2$-norm, i.e. find the coefficients $c_n$ that minimise the quantity

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \Big| f(x) - \sum_{n=-N}^{N} c_n e^{inx}\Big|^2 dx$$

**5.** Assume that $f \in C^2_\#$ and $\int_{-\pi}^{\pi} f(x)dx = 0$. Prove the inequality

$$\int_{-\pi}^{\pi} |f(x)|^2 dx \;\le\; \int_{-\pi}^{\pi} |f''(x)|^2 dx.$$

When do you have equality here?

**6.** Compute the Fourier series of $f(x) = x^2$, $x \in (-\pi,\pi)$ and compute the $L^2$-norm of $f$ in two ways: first by direct computation and then using the Fourier-coefficients. Use this to compute the $\sum_{n=1}^{\infty} n^{-4}$.

**7***[1] Can you compute $\sum_{n=1}^{\infty} n^{-6}$ with the help of Fourier-series?

[1] These ∗-exercises are extras for afficinadoes, not required to get full points from exercises

**Hints for some of the exercises:**

**T.2:** [Hint: Determine $\widehat{f}(n)$ in terms of $\widehat{f'}(n)$, and recall that the Cauchy-Schwarz inequality holds for the inner product $(a,b)_{\ell^2} = \sum_{k=-\infty}^{\infty} a_k \overline{b_k}$ in the space $\ell^2$, see Lecture notes p. 58 (the C-S holds in every inner product space).]

**T.6:** [Recall our result on the equidistribution of $\langle \alpha n \rangle$.]